



FINANCIAL MARKETS WITH SHORT SALES PROHIBITION

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FINANCIAL MARKETS WITH SHORT SALES PROHIBITION

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It is widely thought that short selling practices are a check against speculation and provide hedging mechanisms for many financial investments. Yet, due to its controversial character during economic downturns, regulators have banned short selling in many occasions. In addition, short sales prohibitions are inherent to the majority of emerging markets, commodity markets and the housing market. In this dissertation, we analyze the consequences of short sales prohibition in general semi-martingale financial models. We first prove the Fundamental Theorem of Asset Pricing in continuous time financial models with short sales prohibition and where prices are driven by locally bounded semi-martingales. We then study the theoretical behavior of futures prices in these models. Finally, under our framework, we extend some of the classical results on the hedging problem to general semi-martingale financial models and present a financial connection to the concept of maximal claims.

BIOGRAPHICAL SKETCH

Sergio Andres Pulido Nino was born in Bogota, Colombia, on May 6th of 1982. He obtained a Bachelor in Science, Summa Cum Laude, with a concentration in Mathematics from the Universidad Nacional de Colombia in 2003. In 2005, he earned a Master of Science with a concentration in Mathematics from the Universidad de los Andes. That same year he was admitted to the Ph.D program at Cornell University. He finished his Ph.D studies in 2010. His doctoral research in Financial Mathematics focused on Financial Markets with Short Sales Prohibition. He has accepted a Postdoctoral Associate position in the Department of Mathematical Sciences at Carnegie Mellon University starting September 2010.

To Ana and Annelies

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CHAPTER 1

INTRODUCTION

Short selling has always been a controversial practice and has been alleged to magnify the decline of asset prices. Bans and restrictions on short selling have been commonly used as a regulatory measure to stabilize prices during downturns in the economy. Additionally, only less than half of the more than 150 financial exchanges worldwide allow short sales and the inability to short sell is inherent to specific markets such as commodity markets and the housing market. This dissertation aims to understand the consequences of short sales prohibition in general semi-martingale financial models. The Fundamental Theorem of Asset Pricing establishes the equivalence between the absence of arbitrage, a key concept in mathematical finance, and the existence of a probability measure under which the asset prices in the market have a characteristic behavior. In Chapter 2, we prove the Fundamental Theorem of Asset Pricing in continuous time financial models with short sales prohibition where prices are driven by locally bounded semi-martingales. This extends related results by Jouini and Kallal in [38], Frittelli in [24], Pham and Touzi in [52] and more recently by Karatzas and Kardaras in [41] to the framework of the seminal work of Delbaen and Schachermayer in [12]. Along our presentation, we redefine the concepts of price operator and no dominance and clarify some results obtained by Jarrow, Protter and Shimbo in [36].

To manage risk associated with commitments in markets with short sales constraints, investors substitute spot transactions with trading in futures contracts (see for instance Chapter 3 in [19]). In Chapter 3, we study the behavior of futures prices in markets with short sales prohibition. These results are based

on the Fundamental Theorem of Asset Pricing as stated in Chapter 2. We establish sufficient conditions that guarantee that futures contracts can be used to hedge positions on the spot price processes and present striking mathematical examples when this is not necessarily the case.

Finally, the hedging problem of contingent claims in markets with convex portfolio constraints where prices are driven by diffusions and discrete processes has been extensively studied (see [10], Chapter 5 of [44] and Chapter 9 of [23]). In Chapter 4, inspired by the works of Jacka in [30] and Ansel and Stricker in [2], and using ideas from [22], we extend some of these classical results to general semi-martingale financial models. Additionally, we reveal an interesting financial connection to the concept of maximal claims, first introduced by Delbaen and Schachermayer in [12] and [14].

1.1 Motivation

The current financial crisis, product of the burst of the alleged real estate bubble, has increased the interest of the financial and academic community in the causes and implications of asset price bubbles. In recent works Jarrow, Protter and Shimbo in [36], [37] and Cox and Hobson in [9] developed an arbitrage-free pricing theory for bubbles in complete and incomplete markets. These papers approach the subject by using the insights and tools of mathematical finance, rather than equilibrium arguments where substantial structure, such as investor optimality and market clearing mechanisms, has to be imposed. In their framework, bubbles occur because the market's valuation measure is a local martingale measure which is not a martingale measure and hence the discounted

asset's price is above the expectation of its future cash-flows. The existence of bubbles does not contradict the condition of No Free Lunch with Vanishing Risk (NFLVR), because short sales constraints, given by an admissibility condition on the set of trading strategies, do not allow investors to make a riskless profit from the overpriced securities.

The market model that they considered consists of one risky asset and one riskless bond. The reference filtered probability space, $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, P)$, is assumed to satisfy the usual hypotheses. The price process of the risky asset, $(S_t)_{t \geq 0}$, is a nonnegative semi-martingale on the reference probability space, and the price of the riskless bond B_t is taken constant and equal to 1. The *cumulative cash flows process* of the risky asset is given by a semi-martingale $(D_t)_{t \geq 0}$ with $D_0 = 0$. At a random time τ the asset has a *terminal payoff or liquidation value* of $X_\tau \geq 0$. If $W = S_t 1_{\{t < \tau\}} + D_{t \wedge \tau} + X_\tau 1_{\{t \geq \tau\}}$ is the *wealth process* associated with the market price of the risky asset, the *admissible strategies* in the market are pairs (H, η) with $H \in L(W)$ (integrable with respect to W in the stochastic sense), corresponding to the strategy on the risky asset, and η optional, corresponding to the strategy on the riskless bond, such that

$$HS + \eta = \eta_0 + (H \cdot W) \geq -\alpha,$$

for some $\alpha \geq 0$. Under the assumption that W is locally bounded, The *First Fundamental Theorem of Asset Pricing*, as proven by Delbaen and Schachermayer in [12] implies that the *No Free Lunch with Vanishing Risk* (NFLVR) condition holds if and only if

$$\mathcal{M}_{loc}(W) := \{Q \sim P : W \text{ is a } Q\text{-local martingale}\} \neq \emptyset.$$

The basic idea behind their definition of bubbles is that the mispricing of the risky asset in terms of its future cash flows is due to the fact that the probability

measure that is used for valuation is a *strict local martingale measure* and not a martingale measure. More precisely, they define the *Fundamental Price* by

$$S_t^* = \sum_i E^{Q^i} \left[\int_t^\tau dD_u + X_\tau 1_{\{\tau < \infty\}} \middle| \mathcal{F}_t \right] 1_{\{t < \tau, t \in [\sigma_i, \sigma_{i+1})\}},$$

where the Q^i 's are in $\mathcal{M}_{loc}(W)$ and the σ_i 's are regime shift random times that are fixed from the beginning. There is a bubble at time t if $S_t^* \neq S_t$. Bubbles generally appear when the change of regime generates a change of valuation measure from a martingale measure to a strict local martingale measure.

Jarrow, Protter and Shimbo proved in [36] that if $|\mathcal{M}_{loc}(W)| = 1$ (market completeness), the time τ is bounded and the market satisfies a *no dominance* assumption then the risky asset with price process S_t does not have a bubble. Merton's original definition of no dominance in [48] is the following:

"Security (portfolio) A is dominant over security (portfolio) B, if on some known date in the future, the return on A will exceed the return on B for some possible states of the world, and will be at least as large as on B, in all possible states of the world... A necessary condition for rational option pricing theory is that the option be priced such that it is neither a dominant nor a dominated security".

Since in this market the admissible trading strategies have to be bounded from below, the condition (NFLVR) does not necessarily rule out the possibility of having dominated securities. Heuristically, if A is dominant over B, a trader would take advantage of the situation by shorting B and going long on A. However if the price process of B is unbounded from above this strategy is not admissible and the (NFLVR) condition is not violated.

By restricting the admissible strategies to those that are bounded from below, the condition (NFLVR) is equivalent to the existence of an equivalent local martingale measure which in some cases could be strict and give rise to price bubbles if used for pricing purposes. No dominance represents an additional restriction on the probability measure used for valuation, which in some cases makes bubbles disappear.

Massive short selling is a practice that is often observed after the burst of a price bubble. Examples are the U.S. stock price crash in 1929, the NASDAQ price bubble of 1998-2000 and more recently the housing price bubble. Since the practice of short selling is alleged to magnify the decline of asset prices, it has been banned and restricted many times during history. As such, short sales bans and restrictions have been commonly used as a regulatory measure to stabilize prices during downturns in the economy. The most recent example was in September of 2008 with the prohibition of short selling by the U.S. Securities and Exchange Commission (SEC) for 799 financial companies in an effort to stabilize those companies. At the same time the U.K. Financial Services Authority (FSA) prohibited short selling for 32 financial companies. On September 22, Australia enacted even more extensive measures with a total ban of short selling.

However, short sales prohibitions are seen not only after the burst of a price bubble. In certain cases, the inability to short sell is inherent to the specific market. There are over 150 stock markets worldwide, and thus many are in the third world. In most of the third world emerging markets the practice of short selling is not allowed (see [6]). Additionally in markets such as commodity markets and the housing market primary securities such as mortgages cannot be sold short. This feature is regarded as a source of inefficiency in the market

and motivated the introduction of futures contracts in these markets.

Therefore, it would be interesting to: (i) extend the arbitrage-free pricing theory as presented in the seminal work of Delbaen and Schachermayer in [12] to markets where some of the securities cannot be sold short, and (ii) understand the effect of short sales prohibition on the prices of financial instruments, particularly futures contracts, and on hedging strategies involving these instruments.

CHAPTER 2

THE FUNDAMENTAL THEOREM OF ASSET PRICING

2.1 The set-up

2.1.1 Notation

In this work, unless otherwise specified, we assume that all the random variables and stochastic processes are defined over a filtered space $(\Omega, \mathcal{F}, \mathbb{F})$, where $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ is a filtration of sub-sigma-algebras of \mathcal{F} and T is a fixed finite time horizon. Two probability measures Q and P on (Ω, \mathcal{F}) are equivalent if Q and P have the same sets in \mathcal{F} of probability 0. We write in this case $Q \sim P$. For the probability measures on (Ω, \mathcal{F}) that we consider, we always assume that \mathbb{F} satisfies the usual hypotheses (see p.3 in [53]).

Given a probability measure Q on (Ω, \mathcal{F}) , we identify random variables that are Q -almost surely equal and denote by $L^0(Q)$ the space of equivalence classes of random variables. The space $L^0(Q)$ is equipped with the topology of convergence in Q -probability. For a random variable g that is either Q -almost surely bounded from below or integrable with respect to Q , we denote by $E^Q[g]$ the expectation of g with respect to Q . For $1 \leq p < \infty$, we let $L^p(Q)$ be the space of equivalence classes of random variables f , such that $|f|^p$ is integrable with respect to Q . The space $L^p(Q)$ is equipped with the topology induced by the norm $\|f\|_p := (E^Q[|f|^p])^{\frac{1}{p}}$. The space $L^\infty(Q)$ is the dual space of $L^1(Q)$ consisting of equivalence classes of functions that are essentially bounded. The space $L^\infty(Q)$ can be equipped with the topology induced by the norm $\|f\|_\infty := \text{ess sup } f$

or with the weak-star topology denoted by $\sigma(L^\infty, L^1)$. Observe that if $Q \sim P$, $L^\infty(Q) = L^\infty(P)$ and $L^0(Q) = L^0(P)$, but in general $L^p(Q) \neq L^p(P)$ for $1 \leq p < \infty$. For $p = 0$ or $1 \leq p \leq \infty$ we will denote by $L_+^p(Q)$ the cone of nonnegative random variables in $L^p(Q)$.

We say that an \mathbb{F} -adapted process M is a Q -martingale (respectively Q -supermartingale, Q -submartingale) if $M_t \in L^1(Q)$ for all $t \in [0, T]$ and $E^Q[M_t | \mathcal{F}_s] = M_s$ (resp. $E^Q[M_t | \mathcal{F}_s] \leq M_s$, $E^Q[M_t | \mathcal{F}_s] \geq M_s$) Q -almost surely for all $s < t$ in $[0, T]$. We say that M is a Q -local martingale (respectively Q -local supermartingale, Q -local submartingale) if M has càdlàg paths (continuous from the right with limits from the left) and there is a sequence of stopping times $(T_n)_{n \geq 1}$ such that $Q(T_n = T) \rightarrow 1$ as $n \rightarrow \infty$ and $(M_{t \wedge T_n} 1_{\{T_n > 0\}})_{t \geq 0}$ is a Q -martingale (resp. Q -supermartingale, Q -submartingale) for all n . Analogously, a process X is called a Q -locally bounded process if there is a sequence of stopping times $(T_n)_{n \geq 1}$ such that $Q(T_n = T) \rightarrow 1$ as $n \rightarrow \infty$ and $(M_{t \wedge T_n} 1_{\{T_n > 0\}})_{t \geq 0}$ is uniformly bounded for all n . In general, we say that a process X belongs to a class of processes locally with respect to Q if there is a sequence of stopping times $(T_n)_{n \geq 1}$ such that $Q(T_n = T) \rightarrow 1$ as $n \rightarrow \infty$ and $(X_{t \wedge T_n} 1_{\{T_n > 0\}})_{t \geq 0}$ is in that class for all n . The sigma algebra generated by the left-continuous \mathbb{F} -adapted processes is called the predictable sigma algebra and is denoted by \mathcal{P} . A process is called predictable if it is measurable with respect to \mathcal{P} on $\Omega \times [0, T]$. A process is of finite variation if its paths are of finite variation almost surely. A Q -semi-martingale X is an \mathbb{F} -adapted càdlàg process that can be written as $X = M + A$, where M is a Q -local martingale and A is adapted and of finite variation. When the process A is predictable we say that X is a Q -special, or sometimes simply special semi-martingale. Given a stochastic process X , with paths that have limits from the left, we denote by $\Delta X_t := X_t - X_{t-}$ the jump of X at time t (by convention $X_{0-} = 0$). For a stopping time τ we let X^τ be

the stopped process $X_t^\tau := X_{t \wedge \tau}$. Given a possibly vector-valued semi-martingale X , we will denote by $L(X)$ the space of predictable processes that are integrable (in the stochastic sense) with respect to X . For more details on the definition of the space $L(X)$ and its properties we refer the reader to [32] and Chapter IV of [53]. For $H \in L(X)$ we will denote by $H \cdot S := \int_0^\cdot H_s dX_s$ the stochastic integral of H with respect to X . Given a semi-martingale X , we denote by $\mathcal{E}(X)$ the stochastic exponential of X (see p.84 in [53]). Finally, we denote by $\mathcal{H}^1(Q)$ the set of real valued Q -martingales X , such that $E^Q \left[[X, X]_T^{\frac{1}{2}} \right] < \infty$, where $[X, X]$ is the quadratic variation of X .

2.1.2 The financial market

We focus our analysis on a finite time trading horizon $[0, T]$ and assume that there are N risky assets trading in the market. We suppose, as in the seminal work of Delbaen and Schachermayer in [12], that the price processes of the N risky assets are nonnegative locally bounded P -semi-martingales over a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfies the usual hypotheses. The probability measure P denotes our reference probability measure. We further assume that \mathcal{F}_0 is P -trivial and $\mathcal{F}_T = \mathcal{F}$. Hence, all random variables measurable with respect to \mathcal{F}_0 are P -almost surely constant and there is no additional source of randomness on the probability space other than the one specified by the filtration \mathbb{F} . We denote by $S := (S^i)_{1 \leq i \leq N}$ the \mathbb{R}^N -valued stochastic process representing the prices of the risky assets. Initially, we assume that the spot interest rates are constant and equal to 0, i.e., the price processes are already discounted. We also assume that the risky assets have no cash flows associated to them and there are no transaction costs.

2.1.3 The trading strategies

We fix $0 \leq d \leq N$ and assume that the first d risky assets can be sold short in an admissible fashion to be specified below and that the last $N - d$ risky assets cannot be sold short under any circumstances. This leads us to define the set of admissible strategies in the market as follows.

Definition 2.1. A vector valued process $H = (H^1, \dots, H^N)$, where for $1 \leq i \leq N$ and $t \in [0, T]$ H_t^i denotes the number of shares of asset i held at time t , is called an *admissible trading strategy* if

- (i) $H \in L(S)$,
- (ii) $H_0 = 0$,
- (iii) $(H \cdot S) \geq -\alpha$ for some $\alpha > 0$,
- (iv) $H^i \geq 0$ for all $i > d$.

We let \mathcal{A} be the set of admissible trading strategies.

Hence, we assume that the initial risky assets' holdings are always equal to 0 and therefore initial endowments are always in numéraire denomination. Condition (iii) above is usually called the *admissibility condition* and restricts the agents' strategies to those whose value is uniformly bounded from below over time. The only sources of friction in our market come from conditions (iii) and (iv) above. For every admissible strategy $H \in \mathcal{A}$ we define the optional process H^0 by

$$H^0 := (H \cdot S) - \sum_{i=1}^N H^i S^i. \quad (2.1)$$

If H^0 denotes the balance in the money market account, then the strategy $\bar{H} = (H^0, H)$ is **self-financing** with initial value 0.

2.1.4 No arbitrage conditions

In their seminal works [12] and [16], Delbaen and Schachermayer considered the no arbitrage paradigm known as No Free Lunch with Vanishing Risk (NFLVR) and proved the Fundamental Theorem of Asset Pricing (FTAP) under this framework. The work of Delbaen and Schachermayer improved previous versions of the Fundamental Theorem of Asset Pricing, where the condition of no arbitrage considered was the condition of No Free Lunch (NFL), introduced for the first time by Kreps in [45]. Below we will redefine the above mentioned concepts in our context.

Define the following cones in $L^0(P)$,

$$\mathcal{K} := \{(H \cdot S)_T : H \in \mathcal{A}\}, \quad (2.2)$$

$$C := (\mathcal{K} - L_+^0(P)) \cap L^\infty(P) = \{g \in L^\infty(P) : g = f - h \text{ for some } f \in \mathcal{K} \text{ and } h \in L_+^0(P)\}. \quad (2.3)$$

The cone \mathcal{K} corresponds to the cone of random variables that can be obtained as payoffs of admissible strategies with zero initial endowment. The cone C is the cone of random variables that are P -almost surely bounded and are dominated from above by an element of \mathcal{K} . These sets of random variables are cones and not subspaces of $L^0(P)$ due to conditions (iii) and (iv) in Definition 2.1. We define in our market the following “no arbitrage” type conditions.

Definition 2.2. *We say that the market satisfies the condition of **no arbitrage** (NA) if*

$$C \cap L_+^\infty(P) = \{0\}.$$

Remark 2.3. *Observe that (NA) holds if and only if*

$$\mathcal{K} \cap L_+^0(P) = \{0\}.$$

Indeed, if $C \cap L_+^\infty(P) \neq \{0\}$, then there exist $f \in C \cap L_+^\infty(P)$ and $g \in \mathcal{K}$ such that $f \neq 0$ and $f \leq g$ P -almost surely. This implies that $g \in \mathcal{K} \cap L_+^0(P)$ and $g \neq 0$. Therefore, $\mathcal{K} \cap L_+^0(P) \neq \{0\}$. Conversely, if $\mathcal{K} \cap L_+^0(P) \neq \{0\}$, then there exists $g \neq 0$ in $\mathcal{K} \cap L_+^0(P)$. We have in this case that $g \wedge 1 \in C \cap L_+^\infty(P)$. Because $g \neq 0$, we have that $g \wedge 1 \neq 0$. Therefore, $C \cap L_+^\infty(P) \neq \{0\}$.

In order to prove the Fundamental Theorem of Asset Pricing the condition of (NA) has to be modified. In this regard we have the following definitions.

Definition 2.4. We say that the market satisfies the condition of **No Free Lunch with Vanishing Risk (NFLVR)** if

$$\overline{C} \cap L_+^\infty(P) = \{0\},$$

where the closure above is taken with respect to the $\|\cdot\|_\infty$ norm on $L^\infty(P)$.

Remark 2.5. Observe that (NFLVR) does not hold if and only if there exists a sequence (^nH) in \mathcal{A} , a sequence of bounded random variables (f_n) and a bounded random variable f measurable with respect to \mathcal{F} such that $(^nH \cdot S)_T \geq f_n$ for all n , f_n converges to f in $L^\infty(P)$, $P(f \geq 0) = 1$ and $P(f > 0) > 0$.

Definition 2.6. Similarly we say that the market satisfies the condition of **No Free Lunch (NFL)** if

$$\overline{C}^* \cap L_+^\infty(P) = \{0\},$$

where the closure above is taken with respect to the $\sigma(L^\infty, L^1)$ -topology on $L^\infty(P)$.

It is important to observe that

$$(NFL) \Rightarrow (NFLVR) \Rightarrow (NA).$$

In the next section we prove the Fundamental Theorem of Asset Pricing in our context. This theorem establishes a relationship between the “no arbitrage” type

conditions defined above and the existence of a measure, usually known as the risk neutral measure, under which the price processes behave in a particular way.

2.2 The Fundamental Theorem of Asset Pricing

The results presented in this section are a combination of the results obtained by Frittelli in [24] for simple predictable strategies in markets under convex constraints, and the extension of the classical theorem of Delbaen and Schachermayer (see [12]) to markets with convex cone constraints established by Kabanov in [39]. The characterization of No Free Lunch with Vanishing Risk is in accordance with the Fundamental Theorem of Asset Pricing as proven in [38] by Jouini and Kallal, who assumed that $S_t \in L^2(P)$ for all times t and considered simple predictable strategies.

2.2.1 The set of risk neutral measures

We first define our set of *risk neutral measures*.

Definition 2.7. We let $\mathcal{M}_{sup}(S)$ be the set of probability measures Q on (Ω, \mathcal{F}) such that

- (i) $Q \sim P$ and,
- (ii) For $1 \leq i \leq d$, S^i is a Q -local martingale and, for $d < i \leq N$, S^i is a Q -supermartingale.

We will call the set $\mathcal{M}_{sup}(S)$ the set of *risk neutral measures* or *equivalent supermartingale measures (ESMM)*.

The following proposition plays a crucial role in the analysis below.

Proposition 2.8. *Let C be as in (2.3). Then*

$$\mathcal{M}_{sup}(S) = \{Q \sim P : \sup_{f \in C} E^Q[f] = 0\}.$$

To prove this proposition we need the following results.

Lemma 2.9. *Suppose that Q is a probability measure on (Ω, \mathcal{F}) . Let V be an \mathbb{R}^N -valued Q -semi-martingale such that V^i is Q -local supermartingale for $i > d$, and V^i is a Q -local martingale for $i \leq d$. Let H be an \mathbb{R}^N -valued bounded predictable process, such that $H^i \geq 0$ for $i > d$. Then $(H \cdot V)$ is a Q -local supermartingale.*

Proof. Without loss of generality we can assume that V^i is a Q -supermartingale for $i > d$. Suppose that for $i > d$, $V^i = M^i - A^i$ is the Doob-Meyer decomposition of the Q -supermartingale V^i , with M^i a Q -local martingale and A^i a predictable nondecreasing process such that $A_0^i = 0$. Let $M^i = V^i$ and $A^i = 0$ for $i \leq d$. Then $V = M - A$, with $M = (M^1, \dots, M^N)$ and $A = (A^1, \dots, A^N)$, is the canonical decomposition of the special vector valued semi-martingale V under Q . Since H is bounded, $(H \cdot V)$ is a Q -special semi-martingale, $H \in L(M) \cap L(A)$, $(H \cdot V) = (H \cdot M) - (H \cdot A)$ and $(H \cdot M)$ is a Q -local martingale (see Proposition 2 in [32]). Additionally, since $H^i \geq 0$ for $i > d$ we have that $(H \cdot A)$ is an nondecreasing process starting at 0. We conclude then that $(H \cdot V)$ is a Q -local supermartingale. \square

The following lemma is a known result of stochastic analysis that we present here for completion.

Lemma 2.10. *Suppose that H is a bounded predictable process and $X \in \mathcal{H}^1(Q)$ is a real-valued martingale. Then $H \cdot X$ is also in $\mathcal{H}^1(Q)$. In particular, $H \cdot X$ is a Q -martingale.*

Proof. Assume that $|H| \leq \beta$. We know that $H \cdot X$ is a Q -local martingale (see Theorem IV-29 in [53]). The Burkholder-Davis-Gundy inequalities (Theorem IV-48 in [53]) imply that there exist constants $C_1, C_2 > 0$ such that for all $t \geq 0$

$$\begin{aligned} C_1 E^Q \left[[H \cdot X, H \cdot X]_t^{\frac{1}{2}} \right] &\leq E^Q \left[\sup_{s \leq t} |(H \cdot X)_s| \right] \leq C_2 E^Q \left[[H \cdot X, H \cdot X]_t^{\frac{1}{2}} \right] \\ &= C_2 E^Q \left[(H^2 \cdot [X, X]_t)^{\frac{1}{2}} \right] \\ &\leq \beta C_2 E^Q \left[[X, X]_t^{\frac{1}{2}} \right] < \infty. \end{aligned}$$

□

The next proposition is a key step in the extension of the Fundamental Theorem of Asset Pricing to markets with short sales prohibition and prices driven by arbitrary locally bounded semi-martingales. It extends a well known result of Ansel and Stricker (see Proposition 3.3 in [2]).

Proposition 2.11. *Let $Q \in \mathcal{M}_{sup}(S)$ and $H \in L(S)$ be such that $H^i \geq 0$ for $i > d$. Then, $H \cdot S$ is a Q -local supermartingale if and only if there exists a sequence of stopping times $(T_n)_{n \geq 1}$ that increases Q -almost surely to T and a sequence of nonpositive random variables Θ_n in $L^1(Q)$ such that $\Delta(H \cdot S)^{T_n} = H * \Delta S^{T_n} \geq \Theta_n$ for all n .*

Proof. (\Leftarrow) It is enough to show that for all n , $(H \cdot S)^{T_n}$ is a Q -local supermartingale. Hence, without loss of generality we can assume that $\Delta(H \cdot S) = H * \Delta S \geq \Theta$ with $\Theta \in L^1(Q)$ a nonpositive random variable. By Proposition 3 in [32], if we define

$$U_t = \sum_{s \leq t} 1_{\{|\Delta S_s| > 1 \text{ or } |\Delta(H \cdot S)_s| > 1\}} \Delta S_s$$

there exist a Q -local martingale N and a predictable process of finite variation B such that $H \in L(N) \cap L(B + U)$, $Y := S - U$ is a Q -special semimartingale with bounded jumps and canonical decomposition $Y = N + B$ and $H \cdot N$ is a Q -local martingale. Let $V := B + U$ and $H^\alpha := H1_{\{|H| \leq \alpha\}}$ for $\alpha \geq 0$. We have that $Q \in \mathcal{M}_{sup}(S)$, N is a Q -local martingale and $V = S - N$. This implies that V^i is a Q -local supermartingale for $i > d$, and V^i is a Q -local martingale for $i \leq d$. We can further assume by localization that $N^i \in \mathcal{H}^1(Q)$ for all $i \leq N$ and that V has canonical decomposition $V = M - A$, where M^i in $\mathcal{H}^1(Q)$ and $A^i \geq 0$ is Q -integrable, predictable and nondecreasing for all $i \leq N$ (see Theorem IV-51 in [53]). By Lemmas 2.9 and 2.10, these assumptions imply that for all $\alpha \geq 0$, $H^\alpha \cdot N$ and $H^\alpha \cdot M$ are Q -martingales and $H^\alpha \cdot V$ is a Q -supermartingale. In particular for all stopping times τ , $E^Q[(H^\alpha \cdot N)_\tau] = 0$ and $E^Q[(H^\alpha \cdot V)_\tau] \leq 0$. This implies that for all stopping times τ , $E^Q[|(H \cdot N)_\tau|] = 2E^Q[(H \cdot N)_\tau^-]$ and $E^Q[|(H \cdot V)_\tau|] \leq 2E^Q[(H \cdot V)_\tau^-]$. After these observations, by following the same argument as the one given in the proof of Proposition 3.3 in [2], we find a sequence of stopping times $(\tau_p)_{p \geq 0}$ increasing to T such that $E^Q[|H \cdot V|_{\tau_p}] \leq 12p + 4E^Q[|\Theta|]$ and, for all $\alpha \geq 0$, $|(H^\alpha \cdot V)^{\tau_p}| \leq 4p + |H \cdot V|_{\tau_p}$. An application of the dominated convergence theorem yields that $(H \cdot V)^{\tau_p}$ is a Q -supermartingale for all $p \geq 0$. Since $H \cdot S = H \cdot N + H \cdot V$ and $(H \cdot N)$ is a Q -local martingale, we conclude that $(H \cdot S)$ is a Q -local supermartingale.

(\Rightarrow) The Q -local supermartingale $H \cdot S$ is special. By Proposition 2 in [32], if $S = M - A$ is the canonical decomposition of S with respect to Q , where M^i is a Q -local martingale, $A_0 = 0$ and A^i is an nondecreasing, predictable and Q -locally integrable process for all $i \leq N$, then $H \cdot S = H \cdot M - H \cdot A$ is the canonical decomposition of $H \cdot S$, where $H \cdot M$ is a Q -local martingale and

$H \cdot A$ is nondecreasing, predictable and Q -locally integrable. By Proposition 3.3 in [2] we can find a sequence of stopping times $(T_n)_{n \geq 0}$ that increases to T and a sequence of nonpositive random variables $(\tilde{\Theta}_n)$ in $L^1(Q)$ such that

$$\Delta(H \cdot M)^{T_n} \geq \tilde{\Theta}_n.$$

We can further assume without loss of generality that $(H \cdot A)_{T_n} \in L^1(Q)$ for all n . By taking $\Theta_n = \tilde{\Theta}_n - (H \cdot A)_{T_n}$, we conclude that for all n

$$\Delta(H \cdot S)^{T_n} = \Delta(H \cdot M)^{T_n} - \Delta(H \cdot A)^{T_n} \geq \tilde{\Theta}_n - (H \cdot A)^{T_n} \geq \Theta_n.$$

□

Lemma 2.12. *Let $Q \in \mathcal{M}_{sup}(S)$ and $H \in \mathcal{A}$ (see Definitions 2.1 and 2.7). Then $(H \cdot S)$ is a Q -supermartingale. In particular $(H \cdot S)_T \in L^1(Q)$ and $E^Q[(H \cdot S)_T] \leq 0$.*

Proof. Assume that $(H \cdot S) \geq -\alpha$, with $\alpha \geq 0$. Let $q \geq 0$ be arbitrary. If we define $T_q = \inf\{t \geq 0 : (H \cdot S)_t \geq q - \alpha\}$, we have that $\Delta(H \cdot S)^{T_q} = H * \Delta S^{T_q} \geq -q$. By Proposition 2.11 we conclude that $(H \cdot S)$ is a Q -local supermartingale bounded from below. By Fatou's lemma we obtain that $(H \cdot S)$ is a Q -supermartingale as we wanted to prove. □

Remark 2.13. *This result corresponds to Lemma 2.2 and Proposition 3.1 in [40]. Here we have proved this result by methods similar to the ones appearing in the original proof of Ansel and Stricker in [2]. Additionally, we have given sufficient and necessary conditions for the σ -supermartingale property (see Definition 2.1 in [40]) to hold.*

We are now ready to prove the main proposition of this section.

Proof of Proposition 2.8. By Lemma 2.12

$$\mathcal{M}_{sup}(S) \subset \{Q \sim P : \sup_{f \in \mathcal{C}} E^Q[f] = 0\}.$$

Now suppose that Q is a probability measure equivalent to P such that $E^Q[f] \leq 0$ for all $f \in C$. Fix $1 \leq i \leq N$. Since S^i is locally bounded, there exists a sequence of stopping times (σ_n) increasing to T such that $S^i_{\cdot \wedge \sigma_n}$ is bounded. Let $0 \leq s < t \leq T$, $A \in \mathcal{F}_s$ and $n \geq 0$ be arbitrary. Consider the process $H^i(r, \omega) = 1_A(\omega)1_{(s \wedge \sigma_n, t \wedge \sigma_n]}(r)$. Let $H^j \equiv 0$ for $j \neq i$. We have that $H = (H_1, \dots, H_N) \in \mathcal{A}$, $(H \cdot S)_T \in C$ and

$$0 \geq E^Q[(H \cdot S)_T] = E^Q[1_A(S^i_{t \wedge \sigma_n} - S^i_{s \wedge \sigma_n})].$$

This implies that $S^i_{\cdot \wedge \sigma_n}$ is a Q -supermartingale for all n and S^i is a Q -local supermartingale. Since S^i is nonnegative, by Fatou's lemma we conclude that S^i is a Q -supermartingale. For $1 \leq i \leq d$ we can apply the same argument to the process $H^i(r, \omega) = -1_A(\omega)1_{(s \wedge \sigma_n, t \wedge \sigma_n]}(r)$ to conclude that S^i is a Q -local martingale. Hence

$$\mathcal{M}_{sup}(S) \supset \{Q \sim P : \sup_{f \in C} E^Q[f] = 0\},$$

and the proposition follows. \square

We have seen in the proof of this proposition that the following equality holds.

Corollary 2.14. *Let $\mathcal{M}_{sup}(S)$ be as in Definition 2.7. Then,*

$$\mathcal{M}_{sup}(S) = \{Q \sim P : (H \cdot S) \text{ is a } Q\text{-supermartingale for all } H \in \mathcal{A}\}. \quad (2.4)$$

Remark 2.15. *In [41] the set of measures on the right side of equation (2.4) is also referred as the set of equivalent supermartingale measures. We have proven in Lemma 2.12, that under short sales prohibition, in order to ensure that all the value processes of admissible trading strategies are supermartingales, it is enough to ensure that the prices of the assets that cannot be sold short are supermartingales and the prices of assets that can be admissibly sold short are local martingales. In other words, when we talk about equivalent supermartingale measures, we understand that the **underlying***

price processes, not the value processes, are either supermartingales or local martingales, depending on the restriction of the market.

2.2.2 The main theorem

Proposition 2.8 combined with the Kreps-Yan separation theorem (see Lemma F in [39]) yields the following theorem.

Theorem 2.16. *(NFL) holds if and only if $\mathcal{M}_{sup}(S) \neq \emptyset$.*

Proof. (\Rightarrow) Assume that (NFL) holds. By the Kreps-Yan separation theorem (Lemma F in [39]) there exists a probability measure $Q \sim P$ such that $E^Q[f] \leq 0$ for all $f \in C$. By Proposition 2.8, $Q \in \mathcal{M}_{sup}(S)$.

(\Leftarrow) Suppose that $Q \in \mathcal{M}_{sup}(S)$. Let $Z = \frac{dQ}{dP}$. As shown in Proposition 2.8 we have that

$$C \subset \{g \in L^\infty(P) : E^P[Zg] = E^Q[g] \leq 0\}.$$

The set on the right is closed under the $\sigma(L^\infty, L^1)$ topology on $L^\infty(P)$. Then

$$\begin{aligned} \overline{C}^* \cap L_+^\infty(P) &\subset \{g \in L_+^\infty(P) : E^P[Zg] = E^Q[g] \leq 0\} \\ &= \{0\} \end{aligned}$$

and (NFL) holds. □

The work of Delbaen and Schachermayer in [12], extended later by Kabanov in [39] implies the following surprising result. Karatzas and Kardaras proved a related result in [41]. However, as already explained in Remark 2.15, the set of

equivalent supermartingale measures that they consider is less specific than in our case (in this work the supermartingale and local martingale properties are understood to hold for the underlying price processes).

Theorem 2.17. $(NFLVR) \Leftrightarrow (NFL) \Leftrightarrow \mathcal{M}_{sup}(S) \neq \emptyset$.

In order to prove this theorem we need the following lemma.

Lemma 2.18. $\{(H \cdot S) : H \in \mathcal{A}, (H \cdot S) \geq -1\}$ is a closed subset of the space of vector valued P -semi-martingales on $[0, T]$ with the semi-martingale topology given by the quasinorm

$$D(X) = \sup\{E^P[1 \wedge |(H \cdot X)_T|] : H \text{ predictable and } |H| \leq 1\}. \quad (2.5)$$

Proof. An inspection of the proof of Theorem V.4 in [47], shows that if $(^n H \cdot S) \geq -1$ converges to V in the semi-martingale topology then along a subsequence $^n H$ converges almost surely to $H \in \mathcal{A}$ and $V = (H \cdot S) \geq -1$. \square

Proof of Theorem 2.17. If K_1, K_2 are nonnegative bounded predictable processes, $K_1 K_2 = 0$, $H_1, H_2 \in \mathcal{A}$ are such that $(H_1 \cdot S), (H_2 \cdot S) \geq -1$, and $X := K_1 \cdot (H_1 \cdot S) + K_2 \cdot (H_2 \cdot S) \geq -1$ then associativity of the stochastic integral implies that $X \in \{(H \cdot S) : H \in \mathcal{A}, (H \cdot S) \geq -1\}$. This fact, the lemma above and Theorem 1.2 in [39] imply that (NFLVR) is equivalent to (NFL). \square

This section demonstrates that the results obtained by Jouini and Kallal in [38] and Frittelli in [24], can be extended to a more general model, similar to the one used by Delbaen and Schachermayer in [12]. It is also clear from this characterization that the prices of the risky assets that cannot be sold short could be above its risk-neutral expectation at maturity time, because the condition of

(NFLVR) only guarantees the existence of an equivalent supermartingale measure for those prices.

2.2.3 Non-zero interest rates and cash flows

In this section we will generalize, in a standard manner, the results previously obtained to the case where the riskless bond's price is not constant and the risky assets have a stream of cash flows associated to them. We will assume in what follows that S^0 , the price of the riskless bond, is a positive \mathbb{F} -adapted P -semi-martingale bounded away from 0. We denote by $\tilde{S}^i = (S^0)^{-1}S^i$ the *discounted* price process of asset i for $1 \leq i \leq N$ and by $\tilde{S} := (\tilde{S}^1, \dots, \tilde{S}^N)$ the vector of discounted price processes. We assume that for each asset i , $1 \leq i \leq N$, there exists a cumulative process of cash flows D^i , which is assumed to be an \mathbb{F} -adapted P -semi-martingale. Let $D := (D^1, \dots, D^N)$ be the vector of cumulative cash-flow processes. Also for $1 \leq i \leq N$ we let $M^i := \frac{1}{S^0} \cdot D^i$ (this integral is well defined since S^0 is nonnegative and bounded away from 0). Finally, let $M := (M^1, \dots, M^N)$. Under these hypotheses and notation we extend our definition of admissible strategies as follows.

Definition 2.19. *A vector valued process $H = (H^1, \dots, H^N)$ is called an **admissible trading strategy** if*

- (i) $H \in L(\tilde{S} + M)$.
- (ii) $H_0 = 0$.
- (iii) $(H \cdot (\tilde{S} + M)) \geq -\alpha$ for some $\alpha > 0$.
- (iv) $H^i \geq 0$ for all $i > d$.

We let \mathcal{A} be the set of admissible trading strategies.

We define as before,

$$\mathcal{K} := \{(H \cdot (\tilde{S} + M))_T : H \in \mathcal{A}\}, \quad (2.6)$$

$$C := (\mathcal{K} - L_+^0(P)) \cap L^\infty(P). \quad (2.7)$$

With these sets, the conditions of (NA), (NFL) and (NFLVR) are defined exactly as before. The following theorem is an immediate consequence of Theorem 2.17 and the fact that S^0 is a positive semi-martingale bounded away from 0.

Theorem 2.20. *Under the additional assumption that for all i , M^i is a locally bounded P -semi-martingale, the conditions of (NFLVR) and (NFL) are equivalent to*

$$\mathcal{M}_{sup}(\tilde{S} + M) \neq \emptyset, \quad (2.8)$$

where $\mathcal{M}_{sup}(\tilde{S} + M)$ is the set of probability measures $Q \sim P$ such that $\tilde{S}^i + M^i$ is a Q -local martingale for $1 \leq i \leq d$ and $\tilde{S}^i + M^i$ is a Q -local supermartingale for $d < i \leq N$. If in addition M^i is bounded from below, then $\tilde{S}^i + M^i$ is a Q -supermartingale for Q in $\mathcal{M}_{sup}(\tilde{S} + M)$.

Proof. This corresponds to Theorem 2.17 after replacing S by $\tilde{S} + M$. The only difference is that $\tilde{S} + M$ is not always nonnegative. However, a careful inspection of the proof of Proposition 2.8 shows that the conclusion of this theorem holds. \square

This generalization will allow us to understand the consequences of the introduction of futures contracts that can be sold short in a market with short sales restrictions. We now proceed to characterize the density processes of the risk neutral measures in a market with short sales prohibition.

2.2.4 Density processes of risk neutral measures

For a more detailed discussion of the results presented below we refer the reader to Chapter III of [33]. To simplify our notation, in this section we will assume that $S^0 \equiv 1$ and there are no cash flows. However under mild hypotheses, by the results presented in the previous section (see Theorem 2.20), the analysis below can be extended to markets with stochastic interest rates and assets with cash flows. In order to obtain such an extension, one replaces the underlying price process by the discounted process plus its discounted cash flows.

Let S be an \mathbb{R}^N -valued process representing the prices of the risky assets in the market. Since we have assumed that S is a P -semi-martingale, it has a canonical representation given by

$$S = S_0 + S^c + (x1_{\{|x| \leq 1\}}) * (\mu^S - \nu) + (x1_{\{|x| > 1\}}) * \mu^S + B,$$

where $*$ denotes integration with respect to a random measure (see Section II-1a of [33]) and,

- (i) S^c is a continuous P -local martingale starting at 0, known as **the continuous martingale part of S** ,
- (ii) μ^S is the random measure associated to the jumps of S defined by

$$\mu^S([0, t] \times A) = \sum_{s \leq t} 1_{A \setminus \{0\}}(\Delta S_s),$$

for $0 \leq t \leq T$ and $A \subset \mathbb{R}^N$,

- (iii) ν is the compensator of the random measure μ^S (see Theorem II-1.8 in [33]),
- (iv) B is a predictable \mathbb{R}^N -valued process with components of finite variation.

If we define $C_{i,j} = [(S^c)^i, (S^c)^j]$ then (B, C, ν) are known as the **semi-martingale characteristics of S under P** with respect to the canonical truncation function $h(x) = x1_{\{|x| \leq 1\}}$. According to Proposition II-2..9 in [33] one can find a version of the characteristics (B, C, ν) of S of the form

$$\begin{aligned} B &= b \cdot A, \\ C &= c \cdot A \\ \nu(\omega, dt, dx) &= dA_t(\omega)K_{\omega,t}(dx), \end{aligned} \tag{2.9}$$

where A is a predictable locally integrable nondecreasing process; b and c are predictable processes, with b taking values in \mathbb{R}^N and c taking values in the set of symmetric nonnegative $N \times N$ matrices and $K_{\omega,t}(dx)$ is a transition kernel from $(\Omega \times [0, T], \mathcal{P})$ into $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$ which satisfies

$$\begin{aligned} K_{\omega,t}(\{0\}) &= 0, \quad \int K_{\omega,t}(dx) (|x|^2 \wedge 1) \leq 1, \\ \Delta A_t(\omega) > 0 &\Rightarrow b_t(\omega) = \int K_{\omega,t}(dx) x 1_{\{|x| \leq 1\}}, \\ \Delta A_t(\omega) K_{\omega,t}(\mathbb{R}^N) &\leq 1. \end{aligned} \tag{2.10}$$

Now, given $Q \sim P$, Girsanov's Theorem for semi-martingales (Theorem III-3.24 in [33]) implies that there exists a nonnegative $\tilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^N)$ -measurable function Y (where $\mathcal{B}(\mathbb{R}^N)$ is the Borel sigma-algebra on \mathbb{R}^N and \otimes denotes the product sigma-algebra) and a predictable process β satisfying

$$\begin{aligned} |x 1_{\{|x| \leq 1\}}(Y - 1)| * \nu_t &< \infty \text{ } Q\text{-almost surely for all } t \in [0, T], \\ \left| \sum_{j \leq N} c^{ij} \beta^{ij} \right| \cdot A_t &< \infty \text{ and } \left(\sum_{j,k \leq N} \beta^j c^{jk} \beta^k \right) \cdot A_t < \infty \text{ } Q\text{-almost surely for all } i \text{ and } t \in [0, T], \\ \nu(\omega; \{t\} \times E) = 1 &\Rightarrow \int Y(\omega, t, x) \nu(\omega; \{t\} \times dx) = 1, \end{aligned}$$

and such that the characteristics of S relative to Q are

$$\begin{aligned}\tilde{B}^i &= B^i + \left(\sum_{j \leq N} c^{ij} \beta^j \right) \cdot A + x^i 1_{\{|x| \leq 1\}} (Y - 1) * \nu, \\ \tilde{C} &= C, \\ \tilde{\nu} &= Y \cdot \nu, \text{ where } Y \cdot \nu(\omega; dt, dx) = \nu(\omega; dt, dx) Y(\omega, t, x).\end{aligned}\tag{2.11}$$

Furthermore, according to Lemma III-5.17 in [33], the density process Z of Q relative to P has the form

$$Z = 1 + (Z_- \beta) \cdot S^c + Z_- \left(Y - 1 + \frac{\hat{Y} - a}{1 - a} 1_{\{a < 1\}} \right) * (\mu^S - \nu) + Z' \tag{2.12}$$

where

- (i) Z' is a P -local martingale with $Z'_0 = 0$ and $[(Z')^c, (S^i)^c] = 0$ for all $i \leq N$ and $M_\mu^P[\Delta Z' | \tilde{\mathcal{P}}] = 0$ (see III-3.15 in [33]),

(ii)

$$a_t(\omega) = \nu(\omega; \{t\} \times \mathbb{R}^N),$$

(iii)

$$\hat{Y}_t(\omega) = \begin{cases} \int \nu(\omega; \{t\} \times dx) Y(\omega, t, x) & \text{if this integral converges,} \\ \infty & \text{otherwise.} \end{cases}$$

Taking into consideration the remarks above we have the following result.

Theorem 2.21. *Assume that (B, C, ν) are the semi-martingale characteristics of S relative to P and let (b, c, K, A) be as in (2.9). Assume that $Q \sim P$ and let (Y, β) be as in (2.11) and (2.12), then Q belongs to $\mathcal{M}_{sup}(S)$ if and only if*

(i)

$$b^i + \left(\sum_{j \leq N} c^{ij} \beta^j \right) + \int (x^i (Y - 1_{\{|x| \leq 1\}})) K(dx) = 0$$

$P \otimes A$ -almost surely for $i \leq d$ (where \otimes denotes the product measure) and,

(ii)

$$b^i + \left(\sum_{j \leq N} c^{ij} \beta^j \right) + \int (x^i (Y - 1_{\{|x| \leq 1\}})) K(dx) \leq 0$$

$P \otimes A$ -almost surely for $i > d$.

Proof. (\Rightarrow) Assume that $Q \in \mathcal{M}_{sup}(S)$. In particular, S is a Q -special semi-martingale. According to (2.11) and Proposition II-2.29 in [33] the finite variation predictable part in the decomposition of S^i for $i \leq N$ is given by

$$\left(b^i + \sum_{j \leq N} c^{ij} \beta^j + \int x^i 1_{|x| \leq 1} (Y - 1) K(dx) + \int x^i 1_{\{|x| > 1\}} Y K(dx) \right) \cdot A. \quad (2.13)$$

Since S^i is a Q -local martingale for $i \leq d$, then the process above is 0 P -almost surely for $i \leq d$ and (i) follows. Since S^i is a Q -supermartingale for $i > d$, the process above is nonincreasing for $i > d$ and (ii) follows.

(\Leftarrow) Assume that (i) and (ii) hold. As observed in the proof of Proposition 3.1 in [40] and the proof of Proposition 11.3 in [41], since we are assuming that S^i is nonnegative for all i , conditions (i) and (ii) imply the following integrability condition

$$\int |x| 1_{\{|x| > 1\}} Y K(dx) < \infty.$$

This combined with (i), (ii), the fact that S_0 is constant and observation (2.13) above implies that S^i is a Q -local martingale for $i \leq d$ and S^i is a Q -supermartingale for $i > d$ (see the proofs of Lemma 3.1. in [40] and Proposition 11.3 in [41]). Hence, $Q \in \mathcal{M}_{sup}(S)$.

□

This theorem gives us a complete characterization of the set of measures in $Q \in \mathcal{M}_{sup}(S)$ in terms of the semi-martingale characteristics of the price process

S and the pair (Y, β) appearing in the representation of the density process of Q relative P given by (2.12). It is a crucial result to describe the properties of price processes of financial derivatives in markets with short sales prohibition.

2.2.5 Price Operators, No Dominance and Bubbles

Motivated by the classical approach of Harrison and Kreps in [26] and Harrison and Pliska in [27] we present in this section an equivalent condition to (NFLVR) in terms of the existence of price operators satisfying Merton's no dominance assumption (see p. 143 in [48]) plus additional conditions. We then give the definition of the fundamental price operator and market price operator and define the concept of bubble in this context. For simplicity we will assume in this section that there are no interest rates or cash flows. However, by the results presented in the previous sections, the analysis can be easily generalized to markets with non-zero interest rates and assets with cash flows.

Definition 2.22. A *price operator* is an operator (not necessarily linear)

$$\Lambda_0 : L^\infty(P) \rightarrow \mathbb{R}$$

which is well defined, i.e. if $f, g \in L^\infty(P)$ and $P(f = g) = 1$, then $\Lambda_0(f) = \Lambda_0(g)$.

The domain of a price operator is chosen in order to establish a connection with the FTAP and the condition of (NFLVR). The concept of no dominance proved itself to be of great importance in the work of Jarrow, Protter and Shimbo in [36] and [37]. We redefine this concept in our context.

Definition 2.23. A price operator Λ_0 satisfies the *no dominance condition (ND)* if for all f, g in $L^\infty(P)$ such that $P(f \geq g) = 1$ and $P(f > g) > 0$ we have that $\Lambda_0(f) > \Lambda_0(g)$.

We say that the price operator satisfies the **no dominance condition at 0** $(ND)_0$, if Λ_0 is positive, i.e. for all $f \in L_+^\infty(P)$ with $P(f > 0) > 0$, $\Lambda_0(f) > 0$.

The next result establishes a relationship between the concepts of (ND) and $(NFLVR)$.

Theorem 2.24. *Suppose that there exists a price operator Λ_0 that is lower semicontinuous on $L^\infty(P)$, satisfies $(ND)_0$ and $\Lambda_0(f) \leq 0$ for all $f \in C$ (see (2.3)). Then $(NFLVR)$ holds.*

Proof. Suppose that such a price operator exists and $(NFLVR)$ does not hold. By Remark 2.5, there exists a sequence of elements in C , $(f_n)_{n \geq 0}$, and a random variable f in $L_+^\infty(P)$ such that $f_n \rightarrow f$ in $L^\infty(P)$ and $P(f > 0) > 0$. Our assumptions on Λ_0 imply that

$$0 < \Lambda_0(f) \leq \liminf_n \Lambda_0(f_n) \leq 0,$$

which leads to a contradiction. \square

The following lemmas are immediate consequences of the definition of (ND) .

Lemma 2.25. *If a price operator Λ_0 satisfies the no dominance condition at 0 $(ND)_0$ and is linear then Λ_0 satisfies (ND) .*

Proof. If $P(f \geq g) = 1$ and $P(f > g) > 0$, by the definition of no dominance at 0, $\Lambda_0(f - g) > 0$. Linearity implies that $\Lambda_0(f) > \Lambda_0(g)$. \square

Lemma 2.26. *If a price operator Λ_0 satisfies the no dominance condition at 0 $(ND)_0$ and is linear then Λ_0 is continuous. Moreover, the operator norm $\|\Lambda_0\|$ is equal to $\Lambda_0(1)$.*

Proof. For any $f \in L^\infty(P)$ we have that $P(-\|f\|_\infty \leq f \leq \|f\|_\infty) = 1$. The lemma above and our hypotheses imply that

$$-\|f\|_\infty \Lambda_0(1) = \Lambda_0(-\|f\|_\infty) \leq \Lambda_0(f) \leq \Lambda_0(\|f\|_\infty) = \|f\|_\infty \Lambda_0(1).$$

Hence $|\Lambda_0(f)| \leq \|f\|_\infty \Lambda_0(1)$ ($\Lambda_0(1) > 0$ by the no dominance assumption), the price operator is bounded and therefore continuous. To verify that the operator norm $\|\Lambda_0\|$ is equal to $\Lambda_0(1)$, we apply the operator to the constant function $f \equiv 1$. \square

The next theorem restates the Fundamental Theorem of Asset Pricing in terms of price operators.

Theorem 2.27. *Let \mathcal{L} be the family of price operators Λ_0 such that*

- (i) Λ_0 satisfies $(ND)_0$;
- (ii) $\Lambda_0(f) \leq 0$ for all $f \in C$;
- (iii) Λ_0 is linear with $\Lambda_0(1) = 1$;

Then the equations given by

$$Q(A) = \Lambda_0(1_A), \tag{2.14}$$

$$\Lambda_0(f) = E^Q[f]. \tag{2.15}$$

establish a one-to-one correspondence between \mathcal{L} and $\mathcal{M}_{sup}(S)$ (see Definition 2.7). In particular No Free Lunch with Vanishing Risk (NFLVR) holds if and only if $\mathcal{L} \neq \emptyset$.

Proof. Suppose that $\Lambda_0 \in \mathcal{L}$. Define Q by (2.14). (i) and (iii) imply that Q is a finitely additive positive measure on (Ω, \mathcal{F}_T) with $Q(\Omega) = 1$. The lemma above guarantees the continuity of Λ_0 and hence that Q is σ -additive and a probability measure on (Ω, \mathcal{F}_T) . Condition (i) implies that $Q \sim P$. The definition of the

Lebesgue integral of a nonnegative function, condition (iii) and continuity of Λ_0 imply that for every nonnegative $f \in L_+^\infty(P)$

$$\Lambda_0(f) = E^Q[f].$$

By (iii) we have that for all $f \in L^\infty(P)$

$$\Lambda_0(f) = \Lambda_0(f^+) - \Lambda(f^-) = E^Q[f^+] - E^Q[f^-] = E^Q[f].$$

We conclude that $\Lambda_0(\cdot) = E^Q[\cdot]$, and by using condition (ii) we can prove that $Q \in \mathcal{M}_{sup}(S)$ (see Proposition 2.8). Conversely, if $Q \in \mathcal{M}_{sup}(S)$ it is easy to see that (2.15) defines an element of \mathcal{L} . \square

If we assume that (NFLVR) holds, any price operator of the form $\Lambda_0(\cdot) = E^Q[\cdot]$ with, $Q \in \mathcal{M}_{sup}(S)$, can be naturally extended to $L^1(Q)$. We denote by $\tilde{\Lambda}_0$ this extension. If Q is a strict supermartingale measure for S^i , we have that $\tilde{\Lambda}_0(S_T^i - S_0^i) = E^Q[S_T^i - S_0^i] \neq 0$. In this case the *pricing rule* $\tilde{\Lambda}_0$ does not agree with the market prices. In what follows we fix a measure $Q^* \in \mathcal{M}_{sup}(S)$, and assume that the market chooses this measure for pricing purposes. Observe that $\mathcal{K} \subset L^1(Q^*)$ (see Lemma 2.12) and $S_T^i \in L^1(Q^*)$ for all i . This leads us to the following definitions.

Definition 2.28. *An operator Λ defined on a subspace of $L^0(P)$ that contains $L^1(Q^*)$ is a **market price operator** if*

$$\Lambda(S_T^i - S_0^i) = 0$$

for all i .

Definition 2.29. *The **fundamental price operator** is the price operator (on $L^1(Q^*)$) given by $\Lambda_0^*(\cdot) = E^{Q^*}[\cdot]$.*

Definition 2.30. An element $f \in L^1(Q^*)$ does not have a **bubble** with respect to a market price operator Λ , if $\Lambda(f) = \Lambda_0^*(f)$. When $f = S_T^i - S_0^i$, we simply say that S^i does not have a bubble. In this case, $S_0^i = E^{Q^*}[S_T^i]$ and S^i is a Q^* -martingale.

In complete markets we have the following result proved by Jarrow, Protter and Shimbo in [36]. We give a proof of this result in our context.

Proposition 2.31. Suppose that S has the martingale representation property with respect to Q^* and there exists a sub-linear market price operator Λ such that $\Lambda(f) \leq 0$ for all $f \in \mathcal{K}$ and $\Lambda(a) = a$ for all $a \in \mathbb{R}$, then S^i does not have a bubble for any i .

Proof. By the martingale representation property of S there exists $f \in \mathcal{K}$ such that $S_T^i = E^{Q^*}[S_T^i] + f$. We have that $\Lambda(S_T^i - S_0^i) = 0$. Hence,

$$0 = \Lambda(E^{Q^*}[S_T^i] - S_0^i + f) \leq E^{Q^*}[S_T^i] - S_0^i + \Lambda(f) \leq E^{Q^*}[S_T^i] - S_0^i \leq 0,$$

and the result follows. \square

Motivated by the classical approach to the theory of no arbitrage by Harrison and Kreps in [26] and Harrison and Pliska in [27] and the work of Jarrow, Protter and Shimbo on bubbles in [36] and [37], we have considered a condition slightly stronger than (NFLVR) in terms of the existence of price operators satisfying Merton's no dominance assumption (see Theorem 2.24). We have shown that these conditions are equivalent by adding other hypotheses (see Theorem 2.27). This clarifies the intuition of NFLVR. We have also seen under our set-up, that if the market price operator satisfies certain conditions (see Proposition 2.31), then bubbles do not exist in complete markets, which was a result obtained by Jarrow, Protter and Shimbo in [36].

CHAPTER 3

FUTURES CONTRACTS

In this chapter we will explore the implications that the Fundamental Theorem of Asset Pricing (FTAP) under short sales prohibition (Theorems 2.17 and 2.20) has on futures prices and on hedging strategies that involve these financial instruments. Initially, to simplify our notation, we assume that there is only one risky asset trading in the market and that this asset cannot be sold short under any circumstances. We denote by S the price of the risky asset and call it *the underlying price process or spot price process*. We further assume that there are no cash flows associated to S . As we did in Section 2.2.3, we denote by S^0 the price of the riskless bond, and assume that it is a positive \mathbb{F} -adapted P -semi-martingale bounded away from 0. $\tilde{S} := (S^0)^{-1}S$ corresponds to the *discounted price process* of the risky asset. Our previous analysis has shown that the no arbitrage paradigm of No Free Lunch with Vanishing Risk (NFLVR) guarantees the existence of a probability measure in $\mathcal{M}_{sup}(\tilde{S})$ (see Theorem 2.20). This set of probability measures contains the set of measures $\mathcal{M}_{loc}(\tilde{S})$ defined by

$$\mathcal{M}_{loc}(\tilde{S}) := \{Q \sim P : \tilde{S} \text{ is a } Q\text{-local martingale}\}. \quad (3.1)$$

The Fundamental Theorem of Asset Pricing (FTAP) presented by Delbaen and Schachermayer in [12], shows that $\mathcal{M}_{loc}(\tilde{S}) \neq \emptyset$ if and only if the condition of (NFLVR) holds for admissible trading strategies, without restriction (iv) in Definition 2.19. Hence, by the short sales prohibition on S , the set of *risk neutral* measures is enlarged from $\mathcal{M}_{loc}(\tilde{S})$ to $\mathcal{M}_{sup}(\tilde{S})$. As we did at the end of the previous chapter, we assume that the market chooses a measure $Q^* \in \mathcal{M}_{sup}(\tilde{S})$ for valuation purposes, and we call $\tilde{S}_t^* = E^{Q^*}[\tilde{S}_T | \mathcal{F}_t]$ **the discounted fundamental price of S at time t** . It has been argued that under certain hypotheses on the

agents beliefs in markets with short sales prohibition, the measure Q^* is a strict supermartingale measure, in the sense that $\tilde{S}_0 > \tilde{S}_0^*$. This phenomenon is usually known as the overpricing hypothesis (see for instance [50], Chapter 7 of [21], [4] and [5]). In this case the asset S is said to have a bubble (see Definition 2.30). The case when $Q^* \in \mathcal{M}_{loc}(\tilde{S})$ and \tilde{S} is not a Q^* -martingale (i.e. when Q^* is a *strict local martingale measure*) has been studied extensively (see for instance [13], [9], [46], [36], [20], [37], [51], [49], [55]).

For hedging purposes, it is usually argued that by trading in alternative markets, such as futures markets, the short sales prohibition can be overcome (see Chapter 3 of [21] and Chapter 7 of [19]). In this section we study in detail the consequences of the overpricing phenomenon over the hedging strategies of agents who desire to have a short position on the underlying price process S . We present some examples when the behavior of the futures prices differs radically from that usually seen in markets without short sales restrictions. Our analysis differs from the one used in the bubbles literature in that we consider supermartingale measures, rather than local martingale measures, and the conclusions and examples are of interest for a larger variety of models, including the simplest of them, e.g. the Black-Scholes model and discrete time models. We do not consider any agent preferences in the analysis below and leave for the next chapter the study of the implications of short sales prohibition on the prices of more general types of derivatives.

3.1 Definition

Futures contracts are among the most traded derivatives in financial markets. Because of the efficient transaction mechanisms of the futures markets, futures contracts are often used by investors to replace trading on the spot price process. In this section we explore the possible behavior of futures prices in markets with short sales prohibition and study the consequences that this behavior might have on the hedging strategies used by agents in the market. The most interesting feature about futures contracts is that the stream of cash flows associated to them depends explicitly on the market price operator. We define futures contracts as in [44] and [35].

Definition 3.1. *A futures contract on a risky asset with price process S and maturity time T is a financial instrument with associated stream of cash flows $F_{t,T}$, such that*

- (i) $F_{t,T}$ is a nonnegative \mathbb{F} -adapted P -semi-martingale with $F_{T,T} = S_T$.
- (ii) The market price of the stream of cash flows $(F_{t,T})_t$ is zero at all times.

$F_{t,T}$ is known as the futures price process.

Condition (ii) in the definition above makes the futures price process dependent on the market price operator. It is important to point out that the futures price process is different from the market price of the futures contract which is zero at all times. Investors are allowed to take long and short positions in a futures contract. Intuitively an investor who takes a long position in a futures contract on S at time t , is obligated to purchase the risky asset at maturity time T at a fixed price $F_{t,T}$ specified at time t . The payment is arranged in different installments determined by the fluctuations of the futures prices over the

time horizon. The investor opens a margin account, and when futures prices increase, the increment is deposited in this margin account and when futures prices decrease the negative increment is withdrawn from the margin account. In this way each futures account is said to be marked to market. An investor who takes a short position in a futures contract is entitled to a stream of cash flows opposite to the one of an investor who takes a long position. Investors do not have to pay at the time they enter a long or short position in a futures contract (see (ii) in Definition 3.1). Since futures contracts are synthetic financial products with zero initial cost, taking a short position on a futures contract is the same as selling such a contract, i.e. it is not necessary to locate a lender in order to obtain a stream of cash flows opposite to the one of an investor who has bought a futures contract. All the arrangements of the contract are made thorough a clearing house. Investors who trade in futures markets have margin requirements (the margin account balance has to be at a certain level at all times). We do not consider these margin requirements in our analysis (for a detailed exposition we refer the reader to [19]).

3.2 No arbitrage futures prices

We present below some necessary and sufficient conditions on the futures price process under which the underlying price process S and the futures contract on S with maturity T satisfy the no arbitrage condition of (NFLVR).

Proposition 3.2. *If the futures price process $F_{t,T}$ is a Q -local martingale for some $Q \in \mathcal{M}_{sup}(\tilde{S})$ then the extended market where both the underlying risky asset S and the futures contract on S trade, satisfies the condition of (NFLVR). Conversely, if furthermore S^0 is locally bounded from above, $M := (S^0)^{-1} \cdot F_{\cdot,T}$ is locally bounded, and*

the extended market satisfies (NFLVR) then there exists $Q \in \mathcal{M}_{sup}(\tilde{S})$ such that $F_{t,T}$ is a Q -local martingale.

Proof. (\Rightarrow) We have that $M = (S^0)^{-1} \cdot F_{\cdot,T}$ is a Q -local martingale since $(F_{t,T})_t$ is a Q -martingale and $(S^0)^{-1}$ is bounded (S^0 is bounded away from 0). The conclusion follows from Theorem 2.20.

(\Leftarrow) Assume that S^0 is locally bounded from above and $M = (S^0)^{-1} \cdot F_{\cdot,T}$ is locally bounded. By Theorem 2.20 there exists $Q \in \mathcal{M}_{sup}(\tilde{S})$ such that M is a Q -local martingale, and since $F_{t,T} = F_{0,T} + (S^0 \cdot M)_t$ and S^0 is locally bounded, $F_{\cdot,T}$ is a Q -local martingale as well.

□

Remark 3.3. In the proof above, the process M corresponds to the discounted stream of cash flows of a futures contract on S with maturity T . Notice that in order to have an extended market that satisfies (NFLVR) it suffices to assure the existence of a measure $Q \in \mathcal{M}_{sup}(\tilde{S})$ such that $(F_{t,T})_t$ is a Q -local martingale and not necessarily a Q -martingale (see [35]). Also, since $|\mathcal{M}_{sup}(\tilde{S})| > 1$ this proposition shows that the futures price process is not completely determined by the underlying price process and the arbitrage-free paradigm of (NFLVR). Indeed, for arbitrary $Q \in \mathcal{M}_{sup}(\tilde{S})$

$$F_{t,T} := E^Q[S_T | \mathcal{F}_t],$$

defines the futures price of a futures contract in such a way that the extended market satisfies (NFLVR). In principle these futures price processes could differ across different risk neutral measures (see Example 3.4 below).

Taking into account the remarks made above we will make the following assumption on the market's valuation measure Q^* .

$$\text{The futures price process } F_{\cdot,T} \text{ is a } Q^* \text{-martingale.} \quad (\text{A1})$$

Hence, we will assume that the futures contract on S does not have a bubble (see Definition 2.30).

Example 3.4 (Black-Scholes model under short sales prohibition). *Assume that*

$$dS_t = S_t(\mu dt + \sigma dB_t),$$

where B_t is an \mathbb{F} -Brownian Motion under P and μ and σ are positive constants, i.e.

$$S_t = S_0 \exp\left(\sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right)t\right).$$

Assume that interest rates are identically equal to 0 so $\tilde{S} = S$. Furthermore, suppose that \mathbb{F} is the filtration generated by B . By the martingale representation theorem for Brownian Motion (see Theorem IV-43 in [53]) we have that there exists a predictable process $\tilde{\eta}$ such that for all $t \in [0, T]$,

$$Z_t := E^P\left[\frac{dQ^*}{dP}\Big|\mathcal{F}_t\right] = 1 + (\tilde{\eta} \cdot B)_t. \quad (3.2)$$

Since $Q^* \sim P$, Z is a strictly positive continuous P -martingale, and by letting $\eta = \frac{\tilde{\eta}}{Z}$, we have that

$$Z = \mathcal{E}(\eta \cdot B).$$

Girsanov's theorem (Theorem III-40 in [53]) implies that under Q^* , S has the semimartingale decomposition

$$\begin{aligned} dS_t &= \sigma S_t (dB_t - \eta_t dt) + S_t(\sigma \eta_t + \mu)dt \\ &= \sigma S_t dB_t^* + S_t(\sigma \eta_t + \mu)dt, \end{aligned}$$

where B^* is an \mathbb{F} -Brownian Motion under Q^* . Since S is a Q^* -supermartingale, we conclude that the finite variation process

$$\int_0^\cdot S_s(\sigma \eta_s + \mu) ds,$$

is a Q^* -supermartingale. This implies that the process has to be indistinguishable from a nonincreasing process and $\eta \leq -\frac{\mu}{\sigma}$ $P \otimes \lambda$ -almost surely, where λ denotes the Lebesgue measure on $[0, T]$. Recall that $\frac{\mu}{\sigma}$ is commonly known as the market price of risk. Now, according to Assumption (A1) the futures price process of a futures contract on S with maturity T is given by

$$\begin{aligned} F_{t,T} &= E^{\mathcal{Q}^*}[S_T | \mathcal{F}_t] \\ &= E^{\mathcal{Q}^*}\left[S_0 \exp\left(\sigma B_T^* + \int_0^T \left(\sigma \eta_s + \mu - \frac{\sigma^2}{2}\right) ds\right) \middle| \mathcal{F}_t\right]. \end{aligned}$$

Under the additional assumption that $\int_t^T \sigma \eta_s ds$ is deterministic we would conclude that

$$F_{t,T} = S_0 \exp\left(\int_0^T (\sigma \eta_s + \mu) ds\right) \mathcal{E}(\sigma B^*)_t. \quad (3.3)$$

Recall that under no short sales prohibition and no interest rates, $\eta \equiv -\frac{\mu}{\sigma}$ and the dynamics of the futures price process would be given by

$$S_0 \mathcal{E}(\sigma B^*)_t.$$

Hence, in a market with short sales prohibition, if the overpricing hypothesis holds and the market's pricing measure has additional properties, the futures price process could have an additional discounting factor

$$\exp\left(\int_0^T (\sigma \eta_s + \mu) ds\right).$$

Additionally, it is important to notice that in this case

$$\begin{aligned} \frac{F_{t,T}}{S_t} &= \frac{\exp\left(\int_0^T (\sigma \eta_s + \mu) ds\right) \mathcal{E}(\sigma B^*)_t}{\exp(\mu t) \mathcal{E}(\sigma B)_t} \\ &= \exp\left(\int_t^T (\sigma \eta_s + \mu) ds\right). \end{aligned} \quad (3.4)$$

Since $F_{\cdot,T}$ and S are observable, one could estimate η from market observations. Of course, in order for (3.4) to hold, we are not taking into consideration interest rates (see

Section 3.5 below) and we are making additional assumptions on the process η . These observations agree with empirical evidence on the effect of short sales restrictions on futures prices and stock returns (see for instance [25] and [4]).

3.3 Zero-interest rates

Assume that interest rates are identically equal to 0 so $S = \tilde{S}$. By using the results of Section 2.2.4 we can describe the dynamics of the futures prices under Assumption (A1). Before we do so we establish the following lemmas.

Lemma 3.5. *Suppose that $S > 0$. Then S has one and only one multiplicative decomposition of the form $S = LD$ where L is a positive Q^* -local martingale and D is a positive, predictable and nonincreasing process. Furthermore, if $S = S_0 + N + V$ is the Doob-Meyer decomposition of the Q^* -supermartingale S , with N a Q^* -local martingale and V a predictable and nonincreasing process with $N_0 = V_0 = 0$ then*

$$L = \mathcal{E}\left(\left(\frac{1}{S_- + \Delta V}\right) \cdot N\right), \quad (3.5)$$

$$D = S_0 \left(\mathcal{E}\left(-\left(\frac{1}{S_- + \Delta V}\right) \cdot V\right) \right)^{-1}. \quad (3.6)$$

Proof. The proof of this result can be found in Section VI-2-a of [31]. \square

Lemma 3.6. *Suppose that (B, C, ν) are the semi-martingale characteristics of S under P , let (b, c, K, A) be as in (2.9), and let (Y^*, β^*) be as in (2.11) relative to Q^* , i.e. the density process Z^* of Q^* with respect to P is of the form (2.12), by replacing Z, β and Y by Z^*, β^* and Y^* , respectively. Then the canonical Doob-Meyer decomposition of the Q^* -supermartingale S is $S = S_0 + N + V$ with*

$$N = (S^*)^c + x * (\mu^S - \nu^*), \quad (3.7)$$

$$V = \left(b + c\beta^* + \int x(Y^* - 1_{\{|x| \leq 1\}}) K(dx) \right) \cdot A, \quad (3.8)$$

where $\nu^* = Y^* \cdot \nu$ and $(S^*)^c$ is the continuous martingale part of S relative to Q^* .

Proof. According to (2.11) the semi-martingale characteristics (B^*, C^*, ν^*) of S under Q^* are

$$B^* = B + c\beta^* \cdot A + x 1_{\{|x| \leq 1\}}(Y^* - 1) * \nu,$$

$$C^* = C,$$

$$\nu^* = Y^* \cdot \nu, \text{ where } Y^* \cdot \nu(\omega; dt, dx) = \nu(\omega; dt, dx) Y^*(\omega, t, x).$$

We have seen in the proof of Theorem 2.21 that (3.8) holds. The formula for N follows from the form of the semi-martingale characteristics (B^*, C^*, ν^*) of S under Q^* and Corollary II-2.38 in [33]. \square

By using these two lemmas and the ideas contained in Example 3.4 we have the following result.

Theorem 3.7. *Suppose that $S > 0$ and Assumption (A1) holds. With the notation of Lemma 3.6 if the process*

$$\left(\frac{b + c\beta^* + \int x(Y^* - 1_{\{|x| \leq 1\}}) K(dx)}{S_- + (c\beta^* + \int xY^* K(dx)) \Delta A} \right) \cdot A,$$

is deterministic and

$$L := \mathcal{E} \left(\frac{1}{S_- + (c\beta^* + \int xY^* K(dx)) \Delta A} \cdot ((S^*)^c + x * (\mu^S - \nu^*)) \right)$$

is a true Q^ -martingale then the futures price process $F_{\cdot, T}$ of a futures contract on S with maturity T is given by*

$$F_{t, T} = S_0 \left(\mathcal{E} \left(- \left(\frac{b + c\beta^* + \int x(Y^* - 1_{\{|x| \leq 1\}}) K(dx)}{S_- + (c\beta^* + \int xY^* K(dx)) \Delta A} \right) \cdot A \right) \right)^{-1}_T L_t. \quad (3.9)$$

Proof. This theorem follows from the previous lemmas by writing $F_{t,T} = E^{\mathcal{Q}^*}[S_T|\mathcal{F}_t]$ and observing that by formulas (2.10)

$$\Delta A > 0 \Rightarrow b = \int x 1_{|x| \leq 1} K(dx).$$

□

Remark 3.8. In this result since we are not taking into consideration interest rates, the futures price coincides with the forward price. Recall that the forward price at time t is the value of K such that the price of a forward contract with delivery price K and maturity T has market price 0 at time t . Assuming that there is no bubble for the forward contract this holds if and only if $E^{\mathcal{Q}^*}[S_T - K|\mathcal{F}_t] = 0$ and $K = E^{\mathcal{Q}^*}[S_T|\mathcal{F}_t]$.

Remark 3.9. If we let $X = \left(\frac{b+c\beta^* + \int x(Y^* - 1_{|x| \leq 1}) K(dx)}{S_- + (c\beta^* + \int xY^* K(dx)) \Delta A} \right) \cdot A$, then we have that the factor by which the futures price process and the spot price process differ is equal to

$$\frac{F_{t,T}}{S_t} = \frac{\mathcal{E}(-X)_t}{\mathcal{E}(-X)_T}.$$

When using futures contracts to hedge positions on the spot price process under short sales prohibition it is important to take into account this additional factor.

Example 3.10 (Black-Scholes continued). Observe that in Example 3.4, $dA_t = dt$, $S_- = S$, $b = \mu S$, $c = (\sigma S)^2$, $dS^c = \sigma S dB$, $d(S^*)^c = \sigma S dB^*$, $\beta^* = \frac{\eta}{\sigma S}$, $K \equiv 0$ and formula (3.9) corresponds to (3.3). Indeed in this case

$$\frac{b + c\beta^* + \int x(Y^* - 1_{|x| \leq 1}) K(dx)}{S_- + (c\beta^* + \int xY^* K(dx)) \Delta A} = \frac{\mu S + \sigma \eta S}{S} = \mu + \sigma \eta.$$

We now explore some examples in discrete time when the dynamics of the price process do not resemble those seen in markets with no short sales prohibition.

Example 3.11. Suppose that $\Omega = (\omega_1, \omega_2, \omega_3, \omega_4)$ and $T = 3$. Suppose that the price process of an asset that cannot be sold short is given by S , where $S_t \equiv \frac{21}{16}$ for $0 \leq t < 1$;

$S_t(\omega_1) = S_t(\omega_2) = \frac{11}{8}$, $S_t(\omega_3) = S_t(\omega_4) = \frac{10}{8}$ for $1 \leq t < 2$; $S_t(\omega_1) = \frac{7}{4}$, $S_t(\omega_2) = \frac{1}{4}$, $S_t(\omega_3) = \frac{3}{2}$, $S_t(\omega_4) = \frac{1}{2}$ for $2 \leq t \leq 3$. Let \mathbb{F} be the minimal filtration generated by S . We have in this case that P given by $P(\omega_1) = P(\omega_3) = \frac{3}{8}$, $P(\omega_2) = P(\omega_4) = \frac{1}{8}$ is a martingale-measure for S . Furthermore, P is the only measure that makes S a martingale. Suppose that Q^* is given by $Q^*(\omega_1) = Q^*(\omega_2) = Q^*(\omega_3) = Q^*(\omega_4) = \frac{1}{4}$. Then $Q^* \in \mathcal{M}_{sup}(S)$. We have that $E^{Q^*}[S_t|\mathcal{F}_1] \equiv 1$ for $2 \leq t \leq 3$. It is easy to see that the canonical decomposition of the special semi-martingale S relative to Q^* is $S = S_0 + N + V$, where $V_t \equiv 0$ for $0 \leq t < 2$ and $V_t = 1 - S_1$ for $2 \leq t \leq 3$, $N_t = 0$ for $0 \leq t < 1$, $N_t = S_1 - S_0$ for $1 \leq t < 2$ and $N_t = S_2 + (S_1 - 1) - S_0$ for $2 \leq t \leq 3$. We have that

$$\begin{aligned} \left(\frac{1}{S_- + \Delta V} \cdot N \right)_t &= \frac{S_1 - S_0}{S_0 + 0} 1_{[1,3]}(t) + \frac{S_2 - 1}{S_1 + (1 - S_1)} 1_{[2,3]}(t) \\ &= \frac{S_1 - S_0}{S_0} 1_{[1,3]}(t) + (S_2 - 1) 1_{[2,3]}(t), \end{aligned}$$

$$\begin{aligned} \left(\frac{1}{S_- + \Delta V} \cdot V \right)_t &= \frac{1 - S_1}{S_1 + (1 - S_1)} 1_{[2,3]}(t) \\ &= (1 - S_1) 1_{[2,3]}(t). \end{aligned}$$

The conditions of Theorem 3.7 are not satisfied in this case. Observe that in this case $F_{t,T} \equiv 1$ for $0 \leq t < 2$. The multiplicative decomposition of S (see Lemma 3.5) is given by

$$D = \begin{cases} S_0 & \text{if } 0 \leq t < 2 \\ \frac{S_0}{S_1} & \text{for } 2 \leq t \leq 3, \end{cases}$$

$$L = \begin{cases} 1 & \text{for } 0 \leq t < 1 \\ \frac{S_1}{S_0} & \text{for } 1 \leq t < 2 \\ \frac{S_1 S_2}{S_0} & \text{for } 2 \leq t \leq 3, \end{cases}$$

and formula (3.9) does not hold for $t \in [0, 1)$.

3.4 The representation property

Suppose that the financial market would be complete without any prohibitions on short sales. It is usually believed that, without taking into account interest rates, after introducing futures contracts which can be sold short, the corresponding futures market is complete. In this section we explore sufficient conditions under which this claim holds and provide mathematical counterexamples when it is not the case. Usually the concept of *Market Completeness* is defined in terms of some type of predictable representation property of the assets' price processes (see for instance the seminal works of Harrison and Pliska in [27] and [28]). In this section we discuss some conditions under which such a property on the underlying price process is inherited by the corresponding futures price process. To keep notation simple we assume as before that there is only one underlying price process S trading in the market and that interest rates and associated cash flows are identically zero. We furthermore assume that there exists $P^* \in \mathcal{M}_{loc}(S)$.

Definition 3.12 (Market Completeness). *We say that the financial model is **complete** under P^* if every P^* -local martingale M can be expressed as $M = M_0 + (H \cdot S)$ for some $H \in L(S)$.*

There is a slightly different representation property in terms of the semimartingale characteristics of the price process. This representation property is somehow related to the concept of relaxed completeness as defined in [18], and in general has nothing to do with market completeness (see Proposition 9.4. in [8] and Section 9.5.3. of [7]). The definition below corresponds to Definition III-4.22 in [33].

Definition 3.13. Assume that (B, C, ν) is a triple of semi-martingale characteristics of S under P and S^c is the continuous martingale part of S relative to P . We say that a P -local martingale M has the representation property with respect to S if

$$M = M_0 + H \cdot S^c + W * (\mu^S - \nu),$$

for some $H \in L_{loc}^2(S^c)$ and $W \in G_{loc}(\mu^S)$ (for the definition and properties of the spaces $L_{loc}^2(S^c)$ and $G_{loc}(\mu^S)$ we refer the reader to [33]).

The following result is commonly known as the *Second Fundamental Theorem of Asset Pricing*.

Proposition 3.14. The financial model is complete under P^* if and only if P^* is the only measure equivalent to P that turns S into a local martingale.

Proof. This result essentially corresponds to Corollary 11.4 in [31] (see also [28] and Section 9.5 of [7]). □

Regarding the representation property of Definition 3.13 we have the following analogous proposition.

Proposition 3.15. All P -local martingales have the representation property with respect to S if and only if for every $Q \sim P$ such that S admits the semi-martingale characteristics (B, C, ν) under Q , we have that $Q = P$.

Proof. This result corresponds to Corollary III-4.31 in [33]. □

Furthermore, if the representation property holds for all local martingales with respect to one probability measure then it holds for all local martingales with respect to any equivalent probability measure.

Proposition 3.16. *All P -local martingales have the representation property with respect to S if and only if for all probability measures $Q \sim P$ all Q -local martingales have the representation property with respect to S (relative to the semi-martingale characteristics of S under Q).*

Proof. This corresponds to Theorem III-5.24 in [33]. □

Remark 3.17. *If the hypotheses of Theorem 3.7 hold then the futures price process' dynamics are given by*

$$dF_{t,T} = S_0 D_T dL_t = \frac{S_0 D_T L_{t-}}{S_{t-} + (\Delta V)_t} dN_t,$$

where $S = S_0 + N + V$ is the Doob-Meyer decomposition of the Q^* -supermartingale S . N is the Q^* -local martingale given by

$$N = (S^*)^c + x * (\mu^S - \nu^*),$$

with $(S^*)^c$ the continuous martingale part of S under Q^* and ν^* a “good version” of the compensator of μ^S relative to Q^* . Since $K_t := \frac{S_0 D_T L_{t-}}{S_{t-} + (\Delta V)_t}$ is a locally bounded process (see Section VI-2-a in [31]), by Proposition II-1.30 in [33] we can write

$$dF_{t,T} = K_t d(S^*)^c + d((Kx) * (\mu^S - \nu^*))_t.$$

The proposition above tells us that all P -local martingales have the representation property with respect to S if and only if all Q^* -local martingales M have the form

$$M = M_0 + H \cdot (S^*)^c + W * (\mu^S - \nu^*),$$

for some $H \in L_{loc}^2((S^*)^c)$ and $W \in G_{loc}(\mu^S)$ (here the integrability conditions that define these spaces are taken with respect to Q^*). This representation can be interpreted as a representation form with respect to the futures price process, inherited from the representation property of the spot price process.

When the price processes are continuous we have the following result.

Lemma 3.18. *Suppose that S is a P^* -continuous local martingale. The financial market is complete under P^* if and only if all P -local martingales have the representation property with respect to S .*

Proof. We have that $L_{loc}^2(X) = L(X)$ when X is a continuous P^* -local martingale (see for instance [32]). By Proposition 3.16 all P -local martingales have the representation property with respect to S if and only if all P^* -local martingales have the representation property with respect to S (relative to the P^* -semi-martingale characteristics of S). Since S is the continuous martingale part of S with respect to P^* , the lemma follows. \square

As a consequence we have the following theorem.

Theorem 3.19. *Assume the hypotheses of Theorem 3.7, that S is a continuous process and that the financial market where S trades is complete under P^* . Then, the futures market where a futures contract with futures price process $F_{t,T} = E^{Q^*}[S_T|\mathcal{F}_t]$ trades is complete under Q^* , i.e. all Q^* -local martingales are of the form $x + (H \cdot F_{\cdot,T})$ for some constant x and $H \in L(F_{\cdot,T})$.*

Proof. By the previous lemma, all P -local martingales have the representation property with respect to S . Taking into account the observations made in Remark 3.17 this implies that all Q^* -local martingales M are of the form $M = M_0 + (H \cdot N)$ for some $H \in L_{loc}^2(N)$ (in this case $N = (S^*)^c$). By taking K as in Remark 3.17 (which is a positive process locally bounded and locally bounded away from 0), we have that M is of the form $M = M_0 + (H \cdot N)$ if and only if $M = M_0 + \left(\frac{H}{K} \cdot F_{\cdot,T}\right)$ and the theorem follows. \square

Remark 3.20. *The majority of complete financial models considered in the literature are continuous models or models with price processes driven by compensated Poisson processes. There are however examples of complete financial models with jumps other than models with prices driven by compensated Poisson processes (see for instance [18]). The completeness property of the futures market for models with jumps is a little more delicate. Suppose for instance that the underlying financial market is complete under P^* . Let $Q \sim P^*$ with density of the form $Z^Q = \mathcal{E}(\eta^Q \cdot S)$. Assume that $[S, S]$ and $\int Z_s^Q \eta_s^Q d[S, S]_s$ are Q -locally integrable. Then, the predictable version of Girsanov's theorem (Theorem III-40 in [53]) and Theorem I-3.18 in [33] imply that the predictable part in the canonical decomposition of the Q -special semi-martingale S is given by*

$$\int \eta_s^Q d\langle S, S \rangle_s^Q,$$

where $\langle S, S \rangle^Q$ is the Q -compensator of $[S, S]$. In particular, if Q^ satisfies the above mentioned hypotheses, the Q^* -local martingale part in the canonical decomposition under Q^* is*

$$\tilde{N} = S - \eta^{Q^*} \cdot \langle S, S \rangle^{Q^*}.$$

In this case by uniqueness of the canonical decomposition, for any other measure Q under which \tilde{N} is a Q -local martingale and such that the integrability conditions specified above are satisfied, we have that

$$\eta^{Q^*} \cdot \langle S, S \rangle^{Q^*} = \eta^Q \cdot \langle S, S \rangle^Q.$$

To guarantee completeness of the futures market (under the assumptions of Theorem 3.7), this equation should allow us to conclude that $Z^Q = Z^{Q^}$.*

We have nevertheless the following result analogous to Proposition 3.14 above.

Theorem 3.21. *If Q^* is an extreme point in the set of measures Q absolutely continuous with respect to P such that S is a Q -supermartingale, then under the assumptions of Theorem 3.7 the futures market is complete under Q^* .*

Proof. By Theorem 11.29 in [31], all Q^* -local martingales can be represented in terms of stochastic integrals with respect to N , where N is as in Remark 3.17. The conclusion follows from the fact that under the assumptions of Theorem 3.7 the futures price process is of the form $dF_{t,T} = K_t dN_t$ for a locally bounded process K that is locally bounded away from 0 (see Remark 3.17). \square

The following are examples when some of the conditions in the results above do not hold and the “representation property” of the underlying price process is not inherited by the futures price process.

Example 3.22. *It is easy to check that the model presented in Example 3.11 is complete under P . However, since $F_{\cdot,T}$ is constant in the interval $[0, 2)$ the futures market is not complete under Q^* .*

The example above could be extended to binomial models with independent return jumps.

Example 3.23. *To simplify the notation we present this example in discrete time. It can be extended to a process with jumps in continuous time as in Example 3.11. Suppose that for $t = 1, \dots, T$ the price process is given by $S_t = S_0 \prod_{i=1}^t (1 + R_i)$, where under P , R_1, \dots, R_T are i.i.d random variables with $P(R_1 = r) = \frac{1}{2}$, $P(R_1 = -r) = \frac{1}{2}$, and $0 < r < 1$. It is known that the financial market, where S trades and the filtration considered is the minimal filtration generated by S , is complete under P . Fix $0 < t^* < t$*

and $a < S_{t^*}$. Define a measure Q^* such that for $n > t^*$

$$Q^*(R_n = -r | S_{t^*} = x) = p_x := \frac{(1+r) - \left(\frac{a}{x}\right)^{\frac{1}{T-t^*}}}{2r},$$

$Q^*(R_n = r | S_{t^*} = x) = 1 - p_x$ and Q^* coincides with P over $\sigma(R_1, \dots, R_{t^*})$. If

$$r > 1 - \left(\frac{a}{S_{t^*}}\right)^{\frac{1}{T-t^*}},$$

Q^* is a probability measure equivalent to P such that S is a Q^* -supermartingale. Observe that

$$\begin{aligned} E^{Q^*}[S_T | S_{t^*} = x] &= x \sum_{k=0}^{T-t^*} \binom{T-t^*}{k} (1-r)^k (1+r)^{T-t^*-k} p_x^k (1-p_x)^{T-t^*-k} \\ &= x((1-r)p_x + (1+r)(1-p_x))^{T-t^*} \\ &= x((1+r) - 2rp_x)^{T-t^*} \\ &= a. \end{aligned}$$

Hence, in this case the futures price process $F_{t,T}$ is identically equal to a for $t \leq t^*$ and the futures market is not complete under Q^* .

For continuous processes we have the following example, considered in the work on bubbles by Cox and Hobson in [9].

Example 3.24 (Asset price process with a bubble). Suppose that $P = Q^*$ and for $t < T$

$$S = 1 + \mathcal{E}\left(\int_0^\cdot \frac{dB_s}{\sqrt{T-s}}\right),$$

where B is a P -Brownian motion. We have that if

$$X := \int_0^\cdot \frac{dB_s}{\sqrt{T-s}},$$

then

$$[X]_t = \ln\left(\frac{T}{T-t}\right) \text{ and } \lim_{t \rightarrow T} [X]_t = \infty.$$

Since

$$S_t = 1 + \exp\left(X_t - \frac{1}{2}[X]_t\right) = 1 + \exp\left([X]_t\left(\frac{X_t}{[X]_t} - \frac{1}{2}\right)\right),$$

and $\lim_{t \rightarrow T} \frac{X_t}{[X]_t} = 0$ (see Problem 2.9.3 and Theorem 3.4.6 in [43]), if we define $S_T \equiv 1$, then S is a continuous strict local martingale on $[0, T]$. In this case $F_{t,T} \equiv 1$, and the futures contract could not be used to hedge any risk on S .

In the Brownian framework we can study the dynamics of the futures prices in a more general fashion and we present these considerations in the example below.

Example 3.25. If $dS_t = S_t(\mu dt + \sigma dB_t)$ with B a P -Brownian motion, by following the steps of Example 3.4 we can prove that

$$F_{t,T} = S_0 E^{Q^*} \left[\exp\left(\int_0^T (\mu + \sigma \eta_s) ds\right) \mathcal{E}(\sigma B^*)_T \middle| \mathcal{F}_t \right]$$

where B^* is a Q^* -martingale with respect to \mathbb{F} and η is a predictable process such that $(\mu + \sigma \eta) \leq 0$ $P \otimes \lambda$ -almost surely, where λ is the Lebesgue measure on $[0, T]$. Assume that \mathbb{F} is the minimal filtration generated by B^* (this is a rather delicate assumption, and an interesting discussion on this subject can be found in Chapter V of [54] and Section 5.7.1 of [7]). Assume that $\mu + \sigma \eta - \frac{\sigma^2}{2} = -k(B_t^*)$ for some continuous and nonnegative function k (this hypothesis is also quite delicate since we are imposing a particular functional form dependence of η_t on B_t and η_s for $s < t$). We can write then

$$\begin{aligned} F_{t,T} &= S_0 E^{Q^*} \left[\exp\left(-\int_0^T k(B_s^*) ds\right) \exp(\sigma B_T^*) \middle| \mathcal{F}_t \right] \\ &= S_0 \exp\left(-\int_0^t k(B_s^*) ds\right) E^{Q^*} \left[\exp\left(-\int_t^T k(B_s^*) dt\right) \exp(\sigma B_T^*) \middle| \mathcal{F}_t \right]. \end{aligned}$$

Using the Markovian property of B^* with respect to \mathcal{F}_t and assuming that the hypotheses of the Theorem of Feynman-Kac (Theorem 4.4.2 in [43]) hold then we can write

$$F_{t,T} = S_0 \exp\left(-\int_0^t k(B_s^*) ds\right) v(t, B_t^*),$$

where

$$v(t, x) = E^x \left[\exp \left(- \int_0^{T-t} k(B_s^*) ds \right) \exp(\sigma B_{T-t}^*) \right]$$

solves the Cauchy-problem

$$\begin{aligned} -\frac{\partial v}{\partial t} + kv &= \frac{1}{2} \Delta v; \quad \text{on } [0, T) \times \mathbb{R} \\ v(T, x) &= \exp(\sigma x); \quad x \in \mathbb{R}. \end{aligned}$$

By Itô's formula

$$dF_{t,T} = S_0 \exp \left(- \int_0^t k(B_s^*) ds \right) v_x(t, B_t^*) dB_t^*.$$

We would have in this case that the futures market is complete under P^* (for the original filtration \mathbb{F}) if and only if the process $(v_x(t, B_t^*))_{0 \leq t \leq T}$ is nonzero $P \otimes \lambda$ -almost surely, where λ is the Lebesgue measure on $[0, T]$.

An alternative way to describe the dynamics of the futures price process when the process η is not deterministic uses Malliavin Calculus. Under technical assumptions on η , by the Generalized Clark-Ocone formula (see [42]) we have that

$$dF_{t,T} = \left(E^{\mathcal{Q}^*} [D_t F | \mathcal{F}_t] + E^{\mathcal{Q}^*} \left[F \int_t^T D_t \eta_u dB_u^* \middle| \mathcal{F}_t \right] \right) dB_t^*,$$

where $F = S_T = S_0 \exp(\mu T + \sigma B_T - \frac{\sigma^2}{2} T)$ and D_t denotes the Malliavin derivative. By using properties of the Malliavin derivative it is straightforward to see that

$$D_t F = \sigma F.$$

This implies that

$$dF_{t,T} = \left(\sigma F_{t,T} + E^{\mathcal{Q}^*} \left[S_T \int_t^T D_t \eta_u dB_u^* \middle| \mathcal{F}_t \right] \right) dB_t^*.$$

When η is deterministic this yields

$$dF_{t,T} = \sigma F_{t,T} dB_t^*,$$

and $F_{t,T} = F_{0,T} \mathcal{E}(\sigma B^*)_t = S_0 \exp\left(\int_0^T (\mu + \sigma \eta_s) ds\right) \mathcal{E}(\sigma B^*)_t$, as already observed in Example 3.4. However, when η is random, the volatility of the process $F_{t,T}$ is not necessarily equal to σ and has an additional term equal to

$$E^{Q^*} \left[S_T \int_t^T D_t \eta_u dB_u^* \middle| \mathcal{F}_t \right].$$

This difference in volatility in markets with short sales prohibition was experimentally confirmed in [6]. Also observe that the futures market would be complete if and only if

$$\left(\sigma F_{t,T} + E^{Q^*} \left[S_T \int_t^T D_t \eta_u dB_u^* \middle| \mathcal{F}_t \right] \right) \neq 0$$

Lebesgue-almost everywhere $t \in [0, T]$, P -almost surely.

3.5 Non-zero interest rates

In the previous sections we have considered markets where interest rates are equal to zero and futures and forward prices coincide (see Remark 3.8). Without taking into account short sales restrictions or dividend payments, futures prices differ from spot prices by a factor depending on the interest rates (see for instance [19] and [35]). We proved that under short sales prohibition, when the overpricing hypothesis holds and interest rates are equal to zero, there is a discount factor in the futures price process originated from the multiplicative decomposition of the underlying price process with respect to the market's pricing measure Q^* (see Theorem 3.7). Hence, in a market with short sales prohibition and non-zero interest rates, when the overpricing hypothesis holds and the spot price process has a bubble, the difference between futures prices and spot prices can be expressed as a combination of two factors, the interest rates and the aforementioned “short sales prohibition discount factor”. In this section

we will exhibit explicit formulas that show this difference. Of course when S^0 (the riskless bond's price) is deterministic, the futures prices and forward prices agree (see for instance [19] and [35]) and we have the following extension of Theorem 3.7.

Theorem 3.26. *Suppose that $S > 0$ and Assumption (A1) holds. Suppose that (B, C, ν) are the semi-martingale characteristics of \tilde{S} under P , let (b, c, K, A) be as in (2.9), and let (Y^*, β^*) be as in (2.11) relative to Q^* . Then the canonical Doob-Meyer decomposition of the Q^* -supermartingale \tilde{S} is $\tilde{S} = \tilde{S}_0 + N + V$ with*

$$N := (\tilde{S}^*)^c + x * (\mu^{\tilde{S}} - \nu^*), \quad (3.10)$$

$$V := \left(b + c\beta^* + \int x(Y^* - 1_{|x| \leq 1}) K(dx) \right) \cdot A, \quad (3.11)$$

where $\nu^* = Y^* \cdot \nu$ and $(\tilde{S}^*)^c$ is the continuous martingale part of \tilde{S} relative to Q^* . Furthermore, if the processes S^0 and

$$\left(\frac{b + c\beta^* + \int x(Y^* - 1_{|x| \leq 1}) K(dx)}{\tilde{S}_- + (c\beta^* + \int xY^* K(dx)) \Delta A} \right) \cdot A$$

are deterministic and

$$L := \mathcal{E} \left(\frac{1}{\tilde{S}_- + (c\beta^* + \int xY^* K(dx)) \Delta A} \cdot N \right)$$

is a true Q^* -martingale then the futures price process $F_{\cdot, T}$ of a futures contract on S with maturity T is given by

$$F_{t, T} = \tilde{S}_0 S_T^0 \left(\mathcal{E} \left(- \left(\frac{b + c\beta^* + \int x(Y^* - 1_{|x| \leq 1}) K(dx)}{\tilde{S}_- + (c\beta^* + \int xY^* K(dx)) \Delta A} \right) \cdot A \right)_T \right)^{-1} L_t. \quad (3.12)$$

Proof. This theorem follows directly from Lemma 3.6 and Theorem 3.7 after noticing that

$$F_{t, T} = E^{Q^*}[S_T | \mathcal{F}_t] = S_T^0 E^{Q^*}[\tilde{S}_T | \mathcal{F}_t].$$

□

When S^0 is not deterministic there is an alternative way to represent the futures price process under additional assumptions on the dynamics of the bond's price. We will denote by

$$p(t, T) := E^{\mathcal{Q}^*} [(S_T^0)^{-1} | \mathcal{F}_t], \quad (3.13)$$

the (discounted) price at time t of a zero-coupon bond with maturity T . We have the following alternative characterization of the futures price process

Theorem 3.27. *With the notation of Theorem 3.26 and under Assumption (A1), let*

$$X := \frac{1}{\tilde{S}_- + (c\beta^* + \int xY^*K(dx))\Delta A} \cdot N,$$

$$R := - \left(\frac{b + c\beta^* + \int x(Y^* - 1_{|x| \leq 1})K(dx)}{\tilde{S}_- + (c\beta^* + \int xY^*K(dx))\Delta A} \right) \cdot A,$$

and suppose that

$$p(\cdot, T) = p(0, T)\mathcal{E}(Y),$$

for a \mathcal{Q}^* -local martingale Y with $\Delta Y > -1$. Then, the futures price process is given by

$$F_{t,T} = \frac{\tilde{S}_0}{p(0, T)} E^{\mathcal{Q}^*} \left[\frac{\mathcal{E}(Z)_T}{\mathcal{E}(R)_T} \middle| \mathcal{F}_t \right], \quad (3.14)$$

where

$$Z := X - Y - [X^c - Y^c, Y^c] - \sum_{s \leq \cdot} \left(\Delta(X - Y)_s \frac{\Delta Y_s}{1 + \Delta Y_s} \right),$$

and X^c, Y^c are the continuous parts of the \mathcal{Q}^* -local martingales X and Y , respectively.

Proof. Observe that by Lemma 3.5 and Theorem 3.26

$$\begin{aligned} F_{t,T} &= E^{\mathcal{Q}^*} [S_T | \mathcal{F}_t] \\ &= E^{\mathcal{Q}^*} [\tilde{S}_T (S_T^0) | \mathcal{F}_t] \\ &= E^{\mathcal{Q}^*} \left[\frac{\tilde{S}_T}{p(T, T)} \middle| \mathcal{F}_t \right] \\ &= \frac{\tilde{S}_0}{p(0, T)} E^{\mathcal{Q}^*} \left[\frac{\mathcal{E}(X)_T}{\mathcal{E}(Y)_T \mathcal{E}(R)_T} \middle| \mathcal{F}_t \right]. \end{aligned}$$

The theorem follows from Lemma 3.4 in [41], which shows that

$$\frac{\mathcal{E}(X)}{\mathcal{E}(Y)} = \mathcal{E}(Z).$$

□

Remark 3.28. *The ideas presented in this theorem extend those of Amin and Jarrow in [1]. In their work, there is no short sales prohibition (so $R \equiv 0$), the processes X and Y are continuous, $[X - Y, Y]$ is assumed to be deterministic and $\mathcal{E}(X - Y)$ is a Q^* -martingale. In this case we can rewrite formula (3.14) as*

$$\begin{aligned} F_{t,T} &= \frac{\tilde{S}_0 \exp(-[X - Y, Y]_T)}{p(0, T)} \mathcal{E}(X - Y)_t \\ &= \frac{\tilde{S}_0 \exp(-[X - Y, Y]_T + [X - Y, Y]_t)}{p(0, T)} \mathcal{E}(Z)_t \\ &= \frac{\tilde{S}_t}{p(t, T)} \exp(-[X - Y, Y]_T + [X - Y, Y]_t), \end{aligned}$$

which corresponds to equation (3.26) in [1].

These results exhibit explicitly the fact that futures prices have two sources of randomness, X and Y , one coming from the underlying price process and another from the interest rates Y . Hence, in order to use futures contracts to hedge positions on the spot price, it is not only important to adjust for the “short sales prohibition discount factor” $\mathcal{E}(R)^{-1}$, but also for the interest rate factor by using bonds.

3.6 The multi-dimensional case and futures on an index

In this section we will present a natural extension of the results previously exposed to multidimensional markets. We will also present the possible behavior

of futures on an index under our model, and the effect of short sales prohibition in hedging strategies involving these instruments. We come back in this section to the notation used in Chapter 1. We assume that there are N securities trading in the market, from which the first d assets can be sold short in an admissible way (see Definition 2.1) and the last $N - d$ can not be sold short under any circumstances. Let $S = (S^1, \dots, S^N)$ represent the price processes of these assets. In this section, to simplify our notation we will assume that interest rates are identically equal to zero, however we note that the results can be extended to the case of nonzero interest rates by using the ideas exposed in the previous section. Throughout this section we will assume that the semi-martingale characteristics of S are given by (B, C, ν) , and (b, c, K, A) is a “good version” of these characteristics (see (2.9)). According to Theorem 2.21 and Lemmas 3.5 and 3.6 we can write for each $i \leq N$

$$S^i = D^i L^i,$$

where

$$\begin{aligned} L^i &= \mathcal{E} \left(\left(\frac{1}{S_-^i + \Delta V^i} \right) \cdot N^i \right), \\ D^i &= S_0^i \left(\mathcal{E} \left(- \left(\frac{1}{S_-^i + \Delta V^i} \right) \cdot V^i \right) \right)^{-1}, \\ N^i &= (S^{i,*})^c + x^i * (\mu^{S^i} - \nu^*), \\ V^i &= \left(b^i + \sum_j c^{ij} \beta^{j*} + \int x^i (Y^* - 1_{|x| \leq 1})^i K(dx) \right) \cdot A, \end{aligned}$$

where $\nu^* = Y^* \cdot \nu$ and $(S^{i,*})^c$ is the continuous martingale part of S^i relative to Q^* and (Y^*, β^*) are as in (2.11) relative to Q^* . For each i , the Doob-Meyer decomposition of the Q^* -supermartingale S^i is $S^i = S_0^i + N^i + V^i$. By Theorem 2.21 $V^i \equiv 0$ (resp. V^i is nonincreasing) P -almost surely for $i \leq d$ (resp. $i > d$). For each $i \leq N$ we suppose that Assumption (A1) holds for the futures price of a futures contract on S^i with maturity T . In other words, we assume that if $F_{\cdot, T}^i$ is the futures

price of a futures contract on S^i with maturity T then

$$F_{\cdot,T}^i \text{ is a } Q^*\text{-martingale.} \quad (\text{A2})$$

We denote by $F_{\cdot,T} = (F_{\cdot,T}^1, \dots, F_{\cdot,T}^N)$ the vector of futures price processes. For fixed *deterministic* positive weights ω_i we define the index

$$I = \sum_{i=1}^N \omega_i S^i, \quad (3.15)$$

The following observation immediately follows.

Proposition 3.29. *If I is a Q^* -martingale, then for all $i \leq N$, S^i is a Q^* -martingale.*

Proof. Since S^i is a Q^* -supermartingale for all $i \leq N$ we have that if $E^{Q^*}[S_T^i | \mathcal{F}_t] < S_t^i$ for some $i \leq N$ and $t < T$ then

$$I_t = E^{Q^*}[I_T | \mathcal{F}_t] = \sum_{i \leq N} \omega_i E^{Q^*}[S_T^i | \mathcal{F}_t] < \sum_{i \leq N} \omega_i S_t^i = I_t,$$

which is a contradiction. Then, it must be that S^i is a Q^* -martingale for all $i \leq N$. □

This proposition shows that if the index I could be traded, sold short and did not have a bubble then none of the spot price processes would have bubbles either. However, since indexes are not traded in financial markets but rather futures contracts on indexes, it is fundamental to study the dynamics of the futures prices of a futures contract on I . In this regard we make the following assumption.

The futures price of a futures contract on I , $F_{\cdot,T}^I$, is a Q^* -martingale. (A3)

The following theorem describes the behavior of futures on an index and a hedging result in this context.

Theorem 3.30. Assume that for all $i \leq N$ D^i is deterministic and L^i is a true Q^* -martingale and that Assumptions (A2) and (A3) hold. We have that

(a) For all $i \leq N$, $F_{\cdot, T}^i = D_T^i L^i$ and

$$F^I = \sum_{i \leq N} \omega_i F_{\cdot, T}^i = \sum_{i \leq N} \omega_i D_T^i L^i.$$

(b) Furthermore, if S^i is continuous for all $i \leq N$ and for some probability measure $P^* \sim P$

$$\mathcal{M}_{loc}(S) = \{Q \sim P : S^i \text{ is a } Q\text{-local martingale for all } i \leq N\} = \{P^*\}$$

then,

$$\mathcal{M}_{loc}(F) := \{Q \sim P : F_{\cdot, T}^i \text{ is a } Q\text{-local martingale for all } i \leq N\} = \{Q^*\}.$$

Additionally, any Q^* -local martingale M can be written as

$$M = M_0 + (H \cdot F),$$

for some predictable process $H \in L(F)$, where $L(F)$ is the space of predictable processes integrable with respect to F . If additionally $H^i \in L(F_{\cdot, T}^i)$ for all $i \leq N$ then

$$M = M_0 + (K \cdot Y),$$

where $Y = (F_{\cdot, T}^1, \dots, F_{\cdot, T}^N, F^I)$ and K is a predictable process in $L(Y)$ such that $K^i \geq 0$ for all $i \leq N$.

Proof. (a) This follows directly from the observations made at the beginning of this section and Assumptions (A2) and (A3).

(b) That $\mathcal{M}_{loc}(F) = \{Q^*\}$ and that any Q^* -local martingale M can be written as

$$M = M_0 + (H \cdot F),$$

for some predictable process $H \in L(F)$, follows from the extension to the multidimensional case of Proposition 3.14 and Theorem 3.19 (the same proofs apply to this framework). It remains to prove the representation property with respect to the futures contract on I . Let M be an arbitrary \mathcal{Q}^* -martingale and $H \in L(F)$ such that

$$M = M_0 + (H \cdot M).$$

By part (a), we have that for each $i \leq N$

$$-dF_{t,T}^i = \frac{1}{\omega_i} \left(-dF_t^I + \sum_{j \leq N, j \neq i} \omega_j dF_{t,T}^j \right). \quad (3.16)$$

If additionally $H^i \in L(F_{\cdot,T}^i)$ for all $i \leq N$, we can write (see [32])

$$\begin{aligned} H_t dF_t &= \sum_{i \leq N} H_t^i dF_{t,T}^i \\ &= \sum_{i \leq N} (H_t^i 1_{\{H_t^i \geq 0\}} dF_{t,T}^i + H_t^i 1_{\{H_t^i < 0\}} dF_{t,T}^i) \\ &= \sum_{i \leq N} (H_t^i 1_{\{H_t^i \geq 0\}} dF_{t,T}^i + (-H_t^i) 1_{\{H_t^i < 0\}} (-dF_{t,T}^i)) \\ &= \sum_{i \leq N} \left(H_t^i 1_{\{H_t^i \geq 0\}} dF_{t,T}^i + (-H_t^i) 1_{\{H_t^i < 0\}} \frac{1}{\omega_i} \left(-dF_t^I + \sum_{j \leq N, j \neq i} \omega_j dF_{t,T}^j \right) \right). \end{aligned}$$

Observe that the last equation can be rewritten as

$$H_t dF_t = \sum_{i \leq N} K_t^i dF_{t,T}^i + K_t^{N+1} dF_{t,T}^I,$$

with $K^i \geq 0$ for all $i \leq N$.

□

Remark 3.31. *This theorem tells us that under technical assumptions each hedging strategy involving all the futures contracts on the spot price processes can be replaced,*

by trading on a futures on an index, by a strategy that is long on these futures contracts. It is also important to point out that in order to hedge claims by using a futures contract on an index, one uses the futures contracts on the individual spot price processes, rather than the spot price processes themselves. A short position on $F_{\cdot,T}^I$ combined with a long position on the individual spot price processes (modulus some constant coefficients) is not necessarily equivalent to a short position on the spot price process due to the factors D^i by which spot and futures prices differ (see equation (3.16)).

CHAPTER 4

THE HEDGING PROBLEM AND MAXIMAL CLAIMS

In the previous chapter we studied one particular type of derivative, namely futures contracts, in markets with short sales prohibition. Futures contracts can be used to overcome the trading restriction on the spot price processes. We observed that in order to use futures contracts to hedge positions in these markets, investors should take into account an additional discounting factor originating from the multiplicative decomposition of the underlying price process with respect to the market's pricing measure (see Theorem 3.7). We also gave some mathematical examples when futures contracts cannot be used to hedge risk on the underlying prices (see Section 3.4).

In this chapter we seek not only to explore alternative strategies to overcome the short sales prohibitions, but also to understand the scope of the effects of these restrictions. We study in general semi-martingale financial markets the space of contingent claims that can be super-replicated and perfectly replicated by trading with short sales prohibition. By using the results of Föllmer and Kramkov in [22] we extend the classical results of Ansel and Stricker in [2]. The results presented also extend those in Chapter 5 of [44] and Chapter 9 of [23] to general semi-martingale financial markets. Additionally, we establish, in our context, a connection to the concept of maximal claims as it was first introduced by Delbaen and Schachermayer in [12] and [14]. The Fundamental Theorem of Asset Pricing (Theorem 2.17) can be generalized to the case of convex cone portfolio constraints (see Theorem 4.4 in [41]), and some of the results presented in this chapter could be extended to this framework. However, we specialize to short sales prohibition because in this case the analysis is simplified by the

fact that the set of risk neutral measures is characterized by the behavior of the underlying price processes, rather than the behavior of the value processes of the trading strategies (see Remark 2.15 in Chapter 1). We will assume without loss of generality that the price processes are already discounted and there are no cash flows. We will use the same notation as Chapter 1 as described in Section 2.1. We also recall that $\mathcal{M}_{loc}(S)$ is the set of measures equivalent to P under which S is a local martingale.

4.1 The Hedging Problem

This section shows how the results obtained by Föllmer and Kramkov in [22] extend the usual characterization of attainable claims and claims that can be super-replicated to markets with short sales prohibition. These results extend those presented in Chapter 5 of [44] and Chapter 9 of [23] to general semi-martingale financial models. We will assume that the condition of No Free Lunch with Vanishing Risk (see Theorem 2.17) holds. Recent works (see for instance [29] and [55]) have shown that in order to find suitable trading strategies the condition of (NFLVR) can be weakened and the hedging problem can be studied in markets that admit certain types of arbitrage.

4.1.1 Super-replication

Regarding the super-replication of contingent claims in markets with short sales prohibition we have the following theorem.

Theorem 4.1. *Suppose $\mathcal{M}_{sup}(S) \neq \emptyset$. A nonnegative random variable f measurable*

with respect to \mathcal{F}_T can be written as

$$f = x + (H \cdot S)_T - C_T \quad (4.1)$$

with x constant, $H \in \mathcal{A}$ and $C \geq 0$ an adapted and nondecreasing càdlàg process with $C_0 = 0$ if and only if

$$\sup_{Q \in \mathcal{M}_{sup}(S)} E^Q[f] < \infty.$$

In this case, $x = \sup_{Q \in \mathcal{M}_{sup}(S)} E^Q[f]$ is the minimum amount of initial capital for which there exist $H \in \mathcal{A}$ and $C \geq 0$ an adapted and nondecreasing càdlàg process with $C_0 = 0$ such that (4.1) holds.

Proof. This follows directly from Example 2.2, Example 4.1 and Proposition 4.1 in [22]. \square

Before we give an analogous result regarding perfect replication of contingent claims, we present some examples of contingent claims that cannot be super-replicated under short sales prohibition.

Example 4.2 (Black-Scholes model). Suppose that under P , S is a Geometric Brownian motion with drift μ and volatility σ , i.e. assume that $dS_t = S_t(\mu dt + \sigma dB_t)$ where B is a P -Brownian motion. Let \mathbb{F} be the minimal filtration generated by B that satisfies the usual hypotheses. We know in this case that S is a P^* -martingale where P^* is defined by

$$\frac{dP^*}{dP} = \exp \left\{ -\frac{\mu B_T}{\sigma} - \frac{\mu^2 T}{2\sigma^2} \right\}.$$

If $\gamma \geq \frac{\mu}{\sigma}$ is constant and Q is defined by

$$\frac{dQ}{dP} = \exp \left\{ -\gamma B_T - \frac{\gamma^2 T}{2} \right\},$$

then S is a Q -supermartingale (this is a consequence of Girsanov's theorem, Theorem III-39 in [53]). In this case if we define $f := \frac{1}{S_T}$,

$$\begin{aligned} E^Q[f] &= E^P \left[\frac{1}{S_0} \exp \left\{ (-\sigma - \gamma)B_T - \left(\mu - \frac{\sigma^2}{2} + \frac{\gamma^2}{2} \right) T \right\} \right] \\ &= (1/S_0) \exp \left\{ \frac{(\sigma + \gamma)^2}{2} T - \left(\mu - \frac{\sigma^2}{2} + \frac{\gamma^2}{2} \right) T \right\} \\ &= (1/S_0) \exp\{(\sigma\gamma - \mu + \sigma^2)T\}. \end{aligned}$$

This implies that $\sup_{Q \in \mathcal{M}_{sup}(S)} E^Q[f] = \infty$ and f cannot be super-replicated if S cannot be sold short. In particular, f cannot be perfectly replicated. However, since the unconstrained market is complete under P^* (see Definition 3.12), this claim could be replicated by allowing short selling of the risky asset.

We can generalize the previous example to a more general case.

Example 4.3. This example illustrates how, under certain market hypotheses, it is possible to explicitly exhibit a payoff that cannot be super-replicated without short selling. Suppose that S is of the form $S = \mathcal{E}(R)$. Suppose that R is a continuous P -martingale such that $R_0 = 0$ and $[R, R]_T$ is constant and strictly positive. Let $f = \exp(-R_T)$. We have, by Novikov's criterion (see Theorem III-45 in [53]) that for every $\alpha > 0$, $\frac{dQ^\alpha}{dP} = \mathcal{E}(-\alpha R)_T$ defines a measure $Q^\alpha \in \mathcal{M}_{sup}(S)$. Additionally,

$$\begin{aligned} E^{Q^\alpha}[f] &= E^P[\mathcal{E}(-\alpha R)_T f] \\ &= E^P[\mathcal{E}(-(1 + \alpha)R)_T] \exp((1/2 + \alpha)[R, R]_T) \\ &= \exp((1/2 + \alpha)[R, R]_T) \rightarrow \infty, \end{aligned} \tag{4.2}$$

as α goes to infinity. Hence $\sup_{Q \in \mathcal{M}_{sup}(S)} E^Q[f] = \infty$ and Theorem 4.1 implies that f cannot be super-replicated without selling S short. However, if we assume that market where S can be sold short is complete under P (see Definition 3.12), then in the market

where S can be sold short f can be replicated because it belongs to $L^1(P)$. Indeed, by equation (4.2) we have that

$$E^P[f] = \exp([R, R]_T/2) < \infty.$$

4.1.2 Replication

A question that remains open, however, is whether there exists a characterization of contingent claims that can be perfectly replicated. In this regard we have the following result analogous to the one proven by Ansel and Stricker in [2] (see also Theorems 5.8.1 and 5.8.4 in [44]).

Theorem 4.4. *Suppose $\mathcal{M}_{sup}(S) \neq \emptyset$. For a nonnegative random variable f measurable with respect to \mathcal{F}_T the following statements are equivalent.*

- (i) $f = x + (H \cdot S)_T$ with x constant and $H \in \mathcal{A}$ such that $(H \cdot S)$ is an R^* -martingale for some $R^* \in \mathcal{M}_{sup}(S)$.
- (ii) There exists $R^* \in \mathcal{M}_{sup}(S)$ such that

$$\sup_{Q \in \mathcal{M}_{sup}(S)} E^Q[f] = E^{R^*}[f] < \infty. \quad (4.3)$$

Proof. That (i) implies (ii) follows from the fact that $(H \cdot S)$ is a Q -supermartingale starting at 0 for all $Q \in \mathcal{M}_{sup}(S)$ (see Lemma 2.12). To prove that (ii) implies (i) we define for all t in $[0, T]$

$$V_t := \text{ess sup}_{Q \in \mathcal{M}_{sup}(S)} E^Q[f | \mathcal{F}_t]. \quad (4.4)$$

By Lemma A.1 in [22] the process V is a supermartingale under any $Q \in \mathcal{M}_{sup}(S)$. In particular V is an R^* -supermartingale. The fact that $V_T = f$ and (4.3) imply

that $V_0 = E^{R^*}[V_T]$ and V is a martingale under R^* . On the other hand by Theorem 3.1 in [22], $V = V_0 + (H \cdot S) - C$ for some $H \in \mathcal{A}$ and $C \geq 0$ nondecreasing. Since $(H \cdot S)$ is an R^* -supermartingale (see Lemma 2.12) we conclude that

$$E^{R^*}[C_T] = V_0 + E^{R^*}[(H \cdot S)_T] - E^{R^*}[V_T] \leq 0.$$

Then, $C \equiv 0$ R^* -almost surely and $(H \cdot S)$ is an R^* -martingale. \square

V_t in (4.4) is usually used to define the selling price of the claim f at time t . It represents the minimum cost of super-replication of the claim f at time t (see Proposition 4.1 in [22]). The following proposition gives a particular example of a payoff in markets with continuous price processes which cannot be attained with “martingale strategies”.

Proposition 4.5. *Suppose that the market consists of a single risky asset with continuous price process S . Assume further that S is a P -local martingale which is not constant P -almost surely. Then, $f = 1_{\{S_T \leq S_0\}}$ does not belong to the space*

$$\mathcal{G} := \{x + (H \cdot S)_T : x \in \mathbb{R}, H \in \mathcal{A}, (H \cdot S) \text{ is a } Q\text{-martingale for some } Q \in \mathcal{M}_{sup}(S)\}. \quad (4.5)$$

Proof. For each $n \in \mathbb{N}$, let $(T_{n,m})_m$ be a localizing sequence for $\mathcal{E}(-n(S_t - S_0))$.

Define $Q_{n,m} \in \mathcal{M}_{sup}(S)$ by

$$\frac{dQ_{n,m}}{dP} = \mathcal{E}(-n(S_{T \wedge T_{n,m}} - S_0)).$$

We have that

$$\begin{aligned} E^{Q_{n,m}}[f] &= 1 - E^{Q_{n,m}}[1 - f] \\ &= 1 - E^P \left[1_{\{S_T > S_0\}} \exp \left(-n(S_{T \wedge T_{n,m}} - S_0) - \frac{n^2}{2} [S, S]_{T \wedge T_{n,m}} \right) \right]. \end{aligned}$$

Since the expression under the last expectation is dominated by $\exp(nS_0) \in \mathbb{R}$, the Dominated Convergence Theorem implies that for fixed n

$$\lim_m E^{\mathcal{Q}_{n,m}}[f] = 1 - E^P \left[1_{\{S_T > S_0\}} \exp \left(-n(S_T - S_0) - \frac{n^2}{2} [S, S]_T \right) \right].$$

Applying the Dominated Convergence Theorem once again we obtain that

$$\lim_n \lim_m E^{\mathcal{Q}_{n,m}}[f] = 1.$$

This allows us to conclude that

$$\sup_{Q \in \mathcal{P}(\mathcal{S})} E^Q[f] = 1.$$

However, since f is not P -almost surely constant, this supremum is never attained. The result follows from Theorem 4.4. \square

Remark 4.6. *Moreover, we have proven that in non-trivial markets with continuous price processes, the minimum super-replicating cost of a digital option of the form $1_{\{S_T \leq S_0\}}$ is 1 (See Theorem 4.1). We will give other examples of claims that cannot be perfectly replicated with martingale strategies at the end of this chapter.*

4.1.3 Martingale representation

In this section we make a few remarks about the martingale representation property in markets with short sales constraints. Theorem 4.4 has the following immediate corollary (see Proposition 3.14).

Corollary 4.7. *If $\mathcal{M}_{sup}(S) = \{Q^*\}$ then every Q^* -martingale M with $M_T \geq 0$ P -almost surely is of the form*

$$M = M_0 + (H \cdot S), \tag{4.6}$$

for some $H \in \mathcal{A}$.

Remark 4.8. *This result is closely related to the discussion on completeness presented in Chapter 2. It extends one of the directions of the Second Fundamental Theorem of Asset Pricing (Proposition 3.14). Observe that in general, for $H \in \mathcal{A}$, $(H \cdot S)$ is a Q^* -supermartingale. This result tells us that when Q^* is the only element of $\mathcal{M}_{sup}(S)$, given any Q^* -martingale one can find an appropriate strategy $H \in \mathcal{A}$ that makes $(H \cdot S)$ a Q^* -martingale and (4.6) holds.*

Example 4.9. *Suppose that $S^1 = S^2 = S$ and $\mathcal{M}_{loc}(S) = \{P^*\}$. Assume that S^1 can be sold short in an admissible way but S^2 cannot be sold short under any circumstances. In this case $\mathcal{M}_{sup}(S) = \{P^*\}$. Clearly since S^2 is redundant, the martingale representation property of S^1 implies the representation property of (S^1, S^2) (one simply does not trade S^2).*

Example 4.10. *Suppose for instance that $S = (S^1, \dots, S^N)$ and none of the S^i 's can be sold short. Define an index of the form $I = \sum_i \omega_i S^i$, where the ω_i 's are deterministic and positive. As shown in Proposition 3.29, if I is a Q -martingale for $Q \in \mathcal{M}_{sup}(S)$ then S^i is a Q -martingale for all i . In particular, if for instance I is bounded then one can prove that $\mathcal{M}_{loc}(I) \cap \mathcal{M}_{sup}(S) \subset \mathcal{M}_{loc}(S)$. If $\mathcal{M}_{loc}(S) = \{P^*\}$, then we conclude that in the market where short selling is only prohibited on S , $\mathcal{M}_{sup}(S, I) = \{P^*\}$. By Corollary 4.7, any martingale M is of the form (4.6). This fact can also be proved directly by using Proposition 3.14 and the ideas presented in the proof of Theorem 3.30.*

We now proceed to give an alternative characterization of the random variables in \mathcal{G} , with \mathcal{G} as in (4.5), by extending the concept of maximal claims introduced by Delbaen and Schachermayer in [12] and [14].

4.2 Maximal Claims

By using the extension of the Fundamental Theorem of Asset Pricing presented in Chapter 1, this section generalizes the ideas presented in [14] to markets with short sales prohibition. For simplicity, we assume below that S , the price process of the underlying asset, is one-dimensional. The results can be easily extended to the multi-dimensional case. Recall the definitions of No Arbitrage (NA) and No Free Lunch with Vanishing Risk (NFLVR) given in Chapter 1 and Remark 2.3.

4.2.1 The main theorem

Definition 4.11. Let $\mathcal{B} \subset L^0(P)$. We say that an element f is maximal in \mathcal{B} if

- (i) $f \in \mathcal{B}$ and,
- (ii) $f \leq g$ P -almost surely and $g \in \mathcal{B}$ imply that $f = g$ P -almost surely.

The following is the main theorem of this section.

Theorem 4.12. Let $f \in L^0(P)$ be a random variable bounded from below. The following statements are equivalent.

- (i) $f = (H \cdot S)_T$ for some $H \in \mathcal{A}$ such that
 - (a) the market where $S^1 = (H \cdot S)$ and $S^2 = S$ trade with short selling prohibition on S^2 satisfies (NFLVR) and,

(b) f is maximal in \mathcal{B} where \mathcal{B} is the set of random variables of the form

$$((H^1, H^2) \cdot (S^1, S^2))_T,$$

where $H^2 \geq 0$, $H_0^1 \equiv 1$, $H_0^2 \equiv 0$ and

$$(H^1 - 1, H^2) \cdot (S^1, S^2) \geq -\beta - \alpha S^1 \quad (4.7)$$

for some $\alpha, \beta > 0$.

(ii) There exists $R^* \in \mathcal{M}_{sup}(S)$ such that $\sup_{Q \in \mathcal{M}_{sup}(S)} E^Q[f] = E^{R^*}[f] = 0$.

(iii) $f = (H \cdot S)_T$ for some $H \in \mathcal{A}$ such that $(H \cdot S)$ is an R^* -martingale for some R^* in $\mathcal{M}_{sup}(S)$.

If we further assume that f is bounded and $\mathcal{M}_{loc}(S) \neq \emptyset$, the above statements are equivalent to

(iv) $f = (H \cdot S)_T$ for some $H \in \mathcal{A}$ such that $(H \cdot S)$ is an R -martingale for all R in $\mathcal{M}_{loc}(S)$.

Before establishing some lemmas necessary to prove this theorem we make some remarks.

Remark 4.13. Condition (4.7) resembles the definition of workable claims exposed in [15].

Remark 4.14. If $f = (H \cdot S)_T$, $(H \cdot S)$ is an R^* -martingale for some $R^* \in \mathcal{M}_{sup}(S)$ and $1_{\{H=0\}} \cdot S$ is indistinguishable from 0 then $R^* \in \mathcal{M}_{loc}(S) \neq \emptyset$. Indeed, observe that if we call $M = (H \cdot S)$, then $\left(\frac{1}{H} 1_{\{H \neq 0\}}\right) \cdot M = 1_{\{H \neq 0\}} \cdot S = S - S_0$ is an R^* -local martingale. Theorem 13 in [14] implies that the claim f is also maximal in \mathcal{K} with no short selling prohibition on S . Additionally, also by Theorem 13 in [14], this theorem shows that

when $\mathcal{M}_{loc}(S) \neq \emptyset$, all bounded maximal claims in \mathcal{B} are maximal in \mathcal{K} with no short selling prohibition on S .

Remark 4.15. Suppose that there are no portfolio restrictions other than the admissibility condition (condition (iii) in Definition 2.1). Assume, as in the previous chapters, that the market uses a measure $Q^* \in \mathcal{M}_{loc}(S)$ for valuation purposes. Then it could be the case that $S_0 < E^{Q^*}[S]$, i.e. S has a bubble, however $S_T - S_0$ could be maximal in \mathcal{K} (an example can be found in [17]). We observe then that, $S_T - S_0$ is not maximal in \mathcal{K} if and only if S has a bubble with respect to any risk neutral measure in $\mathcal{M}_{loc}(S)$.

The proof of Theorem 4.12 that we present below mimics the argument presented in [14]. In this generalization the Fundamental Theorem of Asset Pricing under short sales prohibition, Theorem 2.17, and the results presented by Kabanov in [39] are fundamental.

4.2.2 Some lemmas

We first recall the following definition.

Definition 4.16. A subset \mathcal{N} of $L^0(P)$ is bounded in $L^0(P)$ if for all $\epsilon > 0$ there exists $M > 0$ such that $P(|Y| > M) < \epsilon$ for all $Y \in \mathcal{N}$.

The following lemmas will be used.

Lemma 4.17. The condition of (NFLVR) holds if and only if (NA) holds and the set

$$\mathcal{K}_1 = \{(H \cdot S)_T : H \in \mathcal{K} \text{ and } (H \cdot S) \geq -1\}$$

is bounded in $L^0(P)$.

Proof. This corresponds to Lemma 2.2 in [39]. As already noticed before in the proof of Theorem 2.17, the results in [39] can be applied to our case, because the convex portfolio constraints satisfy the desired hypotheses. \square

Lemma 4.18. *The condition of (NFLVR) holds if and only if (NA) holds and there exists a strictly positive P -local martingale $L = (L_t)_{0 \leq t \leq T}$ such that $L_0 = 1$ and $P \in \mathcal{M}_{sup}(LS)$.*

To show this we follow the proof of Theorem 11.2.9 in [14] and observe that it can be extended to our case. For the sake of completion we present the main ideas below.

Proof. (\Rightarrow) If (NFLVR) holds clearly (NA) holds and by Theorem 2.17 there exists Q in $\mathcal{M}_{sup}(S)$. By defining L by $L_t = E^P \left[\frac{dQ}{dP} \middle| \mathcal{F}_t \right]$ we obtain the desired result. Observe that in this case L is not only a P -local martingale but also P -martingale.

(\Leftarrow) Suppose that (NA) holds and there exists a strictly positive P -local martingale L such that $P \in \mathcal{M}_{sup}(LS)$. According to the previous lemma it is enough to show that the set \mathcal{K}_1 is bounded in $L^0(P)$. To prove this we define a sequence of stopping times (T_n) such that L^{T_n} is a martingale for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} P(T_n = T) = 1$. For all $n \in \mathbb{N}$, L^{T_n} defines the density process of a measure in $\mathcal{M}_{sup}(S^{T_n})$. By the previous lemma we have that

$$\mathcal{K}_1^n : \{(H \cdot S)_{T \wedge T_n} : H \text{ is 1-admissible}\}$$

is bounded in $L^0(P)$ for all $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} P(T_n = T) = 1$ we conclude that \mathcal{K}_1 is bounded in $L^0(P)$. Indeed, suppose that \mathcal{K}_1 is not bounded in $L^0(P)$. Then we could find a sequence (H^n) of 1-admissible strategies and $\alpha > 0$ such that $P((H^n \cdot S)_T \geq n) \geq \alpha > 0$. By letting $m \in \mathbb{N}$ be such that $P(T_m < T) < \alpha$ we would

conclude that

$$\begin{aligned}
P((H^n \cdot S)_{T \wedge T_m} \geq n) &\geq P((H^n \cdot S)_T \geq n, T_m = T) \\
&= P((H^n \cdot S)_T \geq n) - P((H^n \cdot S)_T \geq n, T_m < T) \\
&\geq \alpha - P(T_m < T) \\
&> 0.
\end{aligned}$$

This would contradict that \mathcal{K}_1^m is bounded in $L^0(P)$. \square

We now state an analogous result to Theorem 11.4.2 in [14]. This theorem gives necessary and sufficient conditions under which the condition of (NA) holds after a change of numéraire. We will need the following lemma, that proves that the self-financing condition (see (2.1)) is independent of the choice of numéraire (see also [34]).

Lemma 4.19. *Let V be a positive P -semi-martingale, $M = (\frac{S}{V}, \frac{1}{V}, 1)$ and $N = (S, 1, V)$. For a (three-dimensional) predictable process H the following statements are equivalent.*

(i) $H \in L(M)$ and

$$H \cdot M = HM - H_0 M_0 = H^1 \frac{S}{V} + H^2 \frac{1}{V} + H^3 - H_0^1 \frac{S_0}{V_0} - H_0^2 \frac{1}{V_0} - H_0^3,$$

(ii) $H \in L(N)$ and

$$H \cdot N = HN - H_0 N_0 = H^1 S + H^2 + H^3 V - H_0^1 S_0 - H_0^2 - H_0^3 V_0.$$

Proof. (\Rightarrow) Let $W = H \cdot M$. By (i), $\Delta W = H \Delta M = HM - HM_-$ and $W_- = W - \Delta W = HM_- - H_0 M_0$. The integration by parts formula implies that

$$\begin{aligned}
d(VW) &= W_- dV + V_- dW + d[W, V] \\
&= (HM_- - H_0 M_0) dV + V_- H dM + d[W, V].
\end{aligned}$$

Since $d[W, V] = Hd[M, V]$ regrouping terms and using integration by parts once more we obtain that

$$\begin{aligned} d(VW) &= H(M_-dV + V_-dM + d[M, V]) - H_0M_0dV \\ &= Hd(VM) - H_0M_0dV. \end{aligned}$$

We have that $VM = N$, and hence $d(VW) = HdN - H_0M_0dV$. By (i), $VW = HN - VH_0M_0$ and

$$\begin{aligned} HdN &= d(VW) + H_0M_0dV \\ &= (d(HN) - H_0M_0dV) + H_0M_0dV \\ &= d(HN), \end{aligned}$$

as we wanted to show.

(\Leftarrow) The proof of this direction is analogous to the one just presented since M is obtained after multiplying N by the nonnegative semi-martingale $\frac{1}{V}$.

□

Lemma 4.20. *Suppose that V is a strictly positive P -semi-martingale. The market with multidimensional price process $(\frac{1}{V}, \frac{S}{V})$, where short selling prohibition is imposed on $\frac{S}{V}$, satisfies the condition of (NA) if and only if $V_T - V_0$ is maximal in \mathcal{D} , where \mathcal{D} is the set of random variables of the form $(H \cdot (S, V))_T$ where $H^1 \geq 0$, $H_0^1 \equiv 0$, $H_0^2 \equiv 1$ and*

$$(H^1, H^2 - 1) \cdot (S, V) \geq -\alpha V \text{ for some } \alpha > 0.$$

Proof. (\Leftarrow) Let $M = (\frac{1}{V}, \frac{S}{V})$ and $N = (S, V)$. Suppose that $H = (H^1, H^2)$ is an arbitrage in the market with multidimensional price process $(\frac{1}{V}, \frac{S}{V})$. In other words, assume that $H^2 \geq 0$, $H_0 \equiv 0$, $(H \cdot M)_T \geq 0$, $P((H \cdot M)_T > 0) > 0$ and $H \cdot M \geq -\alpha$ for some $\alpha > 0$. If we define

$$H^3 = 1 + H \cdot M - HM,$$

$$\tilde{M} = \left(\frac{1}{V}, \frac{S}{V}, 1 \right),$$

$$\tilde{N} = (1, S, V),$$

and

$$\tilde{H} = (H^1, H^2, H^3)$$

we have that $\tilde{H} \cdot \tilde{M} = \tilde{H}\tilde{M} - 1$. By Lemma 4.19 we have that

$$\tilde{H} \cdot \tilde{N} = \tilde{H}\tilde{N} - V_0.$$

But observe that

$$\tilde{H}\tilde{N} = VHM + (1 + H \cdot M - HM)V = (1 + H \cdot M)V,$$

and,

$$\tilde{H} \cdot \tilde{N} = K \cdot N,$$

where $K = (H^2, H^3)$. Hence $(K \cdot N)_T$ is an element of \mathcal{D} such that $(K \cdot N)_T \geq V_T - V_0$ P -almost surely and $P((K \cdot N)_T > V_T - V_0) > 0$, whence $V_T - V_0$ is not maximal in \mathcal{D} .

(\Rightarrow) Conversely, suppose that $V_T - V_0$ is not maximal in \mathcal{D} . With the notation used above, let $K = (K^1, K^2)$ be a strategy such that $(K \cdot N)_T \geq V_T - V_0$ P -almost surely and $P((K \cdot N)_T > V_T - V_0) > 0$, with $K^1 \geq 0$, $K_0^1 \equiv 0$, $K_0^2 \equiv 1$ and $(K^1, K^2 - 1) \cdot N \geq -\alpha V$ for some $\alpha > 0$. Define $H^2 = K^1$, $H^3 = K^2 - 1$, $H^1 = (H^2, H^3) \cdot N - (H^2, H^3)N$ and $H = (H^1, H^2, H^3)$. We have that $H \cdot \tilde{N} = H\tilde{N} - H_0\tilde{N}_0$. By Lemma 4.19 we have that

$$H \cdot \tilde{M} = H\tilde{M} - H_0\tilde{M}_0 = H\tilde{M}.$$

Hence,

$$(H^1, H^2) \cdot M = H\tilde{M}.$$

We have that

$$H\tilde{M} = \frac{1}{V}H\tilde{N} = \frac{1}{V}((H^2, H^3) \cdot N) = \frac{1}{V}(K \cdot N - (V - V_0)) \geq -\alpha.$$

Therefore,

$$((H^1, H^2) \cdot M)_T = \frac{1}{V_T}((K \cdot N)_T - (V_T - V_0)),$$

$((H^1, H^2) \cdot M)_T \in L_+^0(P)$ and $P(((H^1, H^2) \cdot M)_T > 0) > 0$. Since $H_0^1 = H_0^2 = 0$, (H^1, H^2) is an arbitrage strategy in the market with multi-dimensional price process $(\frac{1}{V}, \frac{S}{V})$.

□

Remark 4.21. *It is important to observe that the no arbitrage condition (NA) over $(\frac{1}{V}, \frac{S}{V})$ holds for strategies that are nonnegative on the second component but can be negative in an admissible way (see condition (iii) in Definition 2.1) over the first component.*

These lemmas allow us to prove Theorem 4.12.

4.2.3 Proof of the main theorem

Proof of Theorem 4.12. • Theorem 4.4 proves the equivalence between (ii) and (iii).

- We will prove now that (iii) implies (i). The Fundamental Theorem of Asset Pricing (Theorem 2.17) shows that (NFLVR) holds for the market consisting of S and $(H \cdot S)$ with short selling prohibition on S . Now assume that $f \leq ((H^1, H^2) \cdot (S^1, S^2))_T$ with $((H^1, H^2) \cdot (S^1, S^2))_T \in \mathcal{B}$. Then

$$(H^1 - 1, H^2) \cdot (S^1, S^2) \geq -\beta - \alpha S^1,$$

for some $\alpha, \beta > 0$ and $((H^1 - 1, H^2) \cdot (S^1, S^2))_T \geq 0$. Since

$$(H^1 - 1 + \alpha, H^2) \cdot (S^1, S^2) \geq -\beta$$

by Lemma 2.12 (extended to the case when the integrand is not identically 0 at time 0) we conclude that

$$(H^1 - 1 + \alpha, H^2) \cdot (S^1, S^2)$$

is an R^* -supermartingale, which in turn implies that $((H^1 - 1, H^2) \cdot (S^1, S^2))$ is an R^* -supermartingale starting at 0. Since $((H^1 - 1, H^2) \cdot (S^1, S^2))_T \geq 0$, we conclude that $((H^1 - 1, H^2) \cdot (S^1, S^2))_T = 0$ P -almost surely. This shows that f is maximal in \mathcal{B} .

- Let us prove now that (i) implies (iii). By the Fundamental Theorem of Asset Pricing we know that there exists $\tilde{P} \in \mathcal{M}_{sup}(S)$ such that $(H \cdot S)$ is a \tilde{P} -local martingale. Let a be such that $V := a + (H \cdot S)$ is positive and bounded away from 0. Since f is maximal in \mathcal{B} , $V - V_0$ is maximal in \mathcal{D} , where \mathcal{D} is as in Lemma 4.20. By Lemma 4.20 (NA) holds in the market where $\frac{S}{V}$ and $\frac{1}{V}$ trade with short selling prohibition on $\frac{S}{V}$. By Lemma 4.18 we conclude that (NFLVR) holds in this market with respect to the measure \tilde{P} . Hence, by the Fundamental Theorem of Asset Pricing there exists $\tilde{Q} \sim \tilde{P}$ (and hence $\tilde{Q} \sim P$) such that $\frac{S}{V}$ is a \tilde{Q} -supermartingale and $\frac{1}{V}$ is a bounded \tilde{Q} -local martingale and therefore a \tilde{Q} -martingale. By defining R^* by $V_T dR^* = d\tilde{Q}$ we observe that $R^* \in \mathcal{M}_{sup}(S)$ and V is a R^* -martingale. This implies that $(H \cdot S)$ is a R^* -martingale as well.

- (iv) Finally to prove that (iii) implies (iv) we observe that if $R \in \mathcal{M}_{loc}(S)$ and (τ_n) is an R -localizing sequence for $(H \cdot S)$ then

$$(H \cdot S)_{\tau_n \wedge T} = E^{R^*}[f | \mathcal{F}_{\tau_n \wedge T}]$$

is a dominated sequence of random variables with zero R -expectation. By the dominated convergence theorem we conclude that $E^R[f] = 0$, and $(H \cdot S)$ is an R -martingale (it is an R -supermartingale with constant expectation).

□

4.2.4 Final remarks

Remark 4.22. Condition (i) in Theorem 4.12 can be interpreted as follows. The market where S^1 and S^2 trade with short sales prohibition on S^2 satisfies the no arbitrage paradigm of (NFLVR). In this market the strategy of buying and holding S^1 cannot be dominated by any strategy with initial holdings of one share of S^1 and none of S^2 that does not sell S^2 short.

The following observation is important. It shows that the elements $f \in L^0(P)$ that satisfy any of the conditions of Theorem 4.12 are maximal in \mathcal{K} .

Proposition 4.23. Condition (ii) (or equivalently Condition (i) or Condition (iii)) in Theorem 4.12 implies that f is maximal in \mathcal{K} .

Proof. Assume that $E^{R^*}[f] = 0$ for some $R^* \in \mathcal{M}_{sup}(S)$. If $f \leq (K \cdot S)_T$ with $K \in \mathcal{A}$, by Lemma 2.12, we conclude that $E^{R^*}[(K \cdot S)_T] = 0$. This implies that $f = (K \cdot S)_T$ P -almost surely and f is maximal in \mathcal{K} . □

Regarding condition (iv) in Theorem 4.12, we recall the following result that gives us alternative conditions under which the value process of the replicating strategy is a martingale.

Corollary 4.24. *Suppose that $f = (H \cdot S)_T$ with $H \in \mathcal{A}$ such that $(H^2 \cdot [S])^{\frac{1}{2}} \in L^1(Q)$ for all $Q \in \mathcal{R}$ where $\emptyset \neq \mathcal{R} \subset \mathcal{M}_{loc}(S)$. Then*

- $(H \cdot S)$ is a Q -martingale for all $Q \in \mathcal{R}$.

Proof. Let $Q \in \mathcal{R}$ be fixed. By the Burkholder-Davis-Gundy Inequalities (Theorem IV-48 in [53]) there exists $C > 0$ such that for all $t \geq 0$

$$E^Q \left[\sup_{s \leq t} |(H \cdot S)_s^*| \right] \leq CE^Q \left[(H^2 \cdot [S])^{\frac{1}{2}} \right] < \infty.$$

We know that $(H \cdot S)$ is an R -local martingale (see [2]). Theorem I-51 in [53] implies that $(H \cdot S)$ is a Q -martingale. \square

Remark 4.25. *A related result for diffusion price processes can be found in Theorem 5.8.4 in [44]. This theorem uses the alternative assumption that*

$$\{(H \cdot S)_\rho : \rho \text{ is a stopping time in } [0, T]\}$$

is Q -uniformly integrable for all $Q \in \mathcal{M}_{sup}(S)$. This hypothesis also implies that $(H \cdot S)$ is a Q -martingale for $Q \in \mathcal{M}_{loc}(S)$.

It is important to point out that in general, the conclusion of (iv) does not hold. An example of such a market can be found in [17]. Theorem 4.12 is useful to argue why certain types of contingent claims in certain financial models cannot be replicated by using a strategy that is maximal in the sense of (i) of Theorem 4.12 above.

Example 4.26. *Let $K \in [0, \infty]$ be fixed. Assume that S is a continuous P -martingale, $[S]$ is deterministic and $P(S_T < K, \tau < T) > 0$ where*

$$\tau = \inf \left\{ t \leq T : S_t \geq K + \frac{1}{2} ([S]_T - [S]_t) \right\} \wedge T.$$

By Novikov's criterion (Theorem III-45 in [53]) we know that

$$\frac{dQ}{dP} = \mathcal{E}\left(-\int_0^T 1_{[\tau, T]}(s) dS_s\right),$$

defines a probability measure $Q \in \mathcal{M}_{sup}(S)$. If $g : [0, \infty) \rightarrow [0, \infty)$ is a function that vanishes on $[K, \infty)$ and is strictly positive on $[0, K)$, then

$$\begin{aligned} E^Q[g(S_T)] &= E^Q[g(S_T)1_{\{S_T < K\}}] \\ &\geq E^P[1_{\{\tau=T\}}g(S_T)1_{\{S_T < K\}}] + E^P[1_{\{S_T < K, \tau < T\}}g(S_T)\exp(-(S_T - K))] \\ &> E^P[g(S_T)]. \end{aligned}$$

If we further assume that g is bounded, then by Theorem 4.12 (condition (iv)) we conclude that $g(S_T)$ does not belong to \mathcal{G}

$$\mathcal{G} = \{x + (H \cdot S)_T : x \in \mathbb{R}, H \in \mathcal{A}, (H \cdot S) \text{ is a } Q\text{-martingale for some } Q \in \mathcal{M}_{sup}(S)\}. \quad (4.8)$$

The function $g(x) = (K - x)_+$ satisfies the above mentioned conditions. Hence under these assumptions, the put option's payoff does not belong to \mathcal{G} .

Remark 4.27. In Example 5.7.4 in [44] and Section 8.1 in [11], it is proven that for diffusion models with constant coefficients and stochastic volatility models with additional properties, respectively, the minimum super-replication price of an European put option $\sup_{Q \in \mathcal{M}_{sup}(S)} E^Q[(K - S_T)_+]$ is equal to K . In particular if $P(S_T \neq K) > 0$, then this supremum is never attained and $(K - S_T)_+$ is not in \mathcal{G} as defined by (4.8).

We will finish this chapter by making some additional remarks on the price of calls and puts in markets with short sales prohibition.

Remark 4.28 (Call and Put Options). Assume as in the previous chapter that the market's valuation measure is $Q^* \in \mathcal{M}_{sup}(S)$. Then for any $K > 0$

$$E^{Q^*}[(K - S_T)_+] - E^{Q^*}[(S_T - K)_+] = (K - E^{Q^*}[S_T]) \geq (K - S_0),$$

where we have equality if and only if S is a Q^* -martingale. This shows that under short sales prohibition a strategy that is long on the put and short on the call might not perfectly replicate a short position on the underlying. Also in the case of short sales prohibition the usual argument that shows that (without considering any dividend payments) the price of an European Call Option is equal to the price of an American Call Option does not necessarily carry out ($(K - S)_+$ is not necessarily a Q^* -sub-martingale). This proves mathematically the empirical observations in [3], which studies markets where it is hard to borrow stock.

In this chapter we have studied the space of contingent claims that can be super-replicated and perfectly replicated with martingale strategies in a market with short sales prohibition. We extended results found in [2],[44] and [23] to the short sales prohibition case. We additionally have extended the results in [14] to our framework and modified the concept of maximality accordingly (see Theorem 4.12). Additionally, we also exposed explicit payoffs in general markets that cannot be replicated without selling the spot price process short.

CHAPTER 5

SUMMARY AND FUTURE RESEARCH

In Chapter 2, Theorems 2.17 and 2.20, we proved that the no arbitrage paradigm of No Free Lunch with Vanishing Risk extended to markets with short sales prohibition is equivalent to the existence of an equivalent probability measure that is a local martingale measure for the assets that can be admissibly sold short (see (iii) in Definitions 2.1 and 2.19) and a supermartingale measure for the assets that can never be sold short. This extends the seminal work of Delbaen and Schachermayer in [12] to the context of short sales prohibition and builds on the exposition of Kabanov in [39]. The results are in accordance with previous studies on markets with portfolio constraints by Jouini and Kallal in [38], Frittelli in [24], Pham and Touzi in [52] and more recently by Karatzas and Kardaras in [41] (see Remark 2.15). Proposition 2.11 and Lemma 2.12 are fundamental in the proof and generalize previous results obtained by Ansel and Stricker in [2] and more recently by Kallsen in [40]. In Section 2.2.4, by using results found in Chapter III of [33], in [40] and in [41], we gave a complete characterization of the set of risk neutral measures in markets with short sales prohibition. Finally, at the end of Chapter 2, motivated by the seminal works of Harrison and Kreps in [26] and Harrison and Pliska in [27], and more recently by the works on bubbles by Jarrow, Protter and Shimbo in [36] and [37], we redefined the concepts of Price Operator, No Dominance and Bubble and clarified the relationship between NFLVR and ND (see Theorem 2.27) and the conditions that assure the non-existence of bubbles in complete markets (see Proposition 2.31).

In Chapter 3, we described the dynamics of futures price processes that are coherent with the no arbitrage condition of NFLVR (see Theorems 3.7, 3.26 and

3.27). We observed that in markets with short sales constraints, the difference between futures price processes and underlying price processes could come from two different factors: the risk-free interest rates and an additional factor originated from the Doob-Meyer decomposition of the spot price process with respect to the market's valuation measure (see Example 3.4 and Remark 3.9). We also presented sufficient conditions under which the representation property of the spot price processes is inherited by the futures price processes and show examples when this is not the case (see Section 3.4). At the end of Chapter 3, we explained how the introduction of futures contracts on an index could be used for hedging purposes in markets with short sales prohibition (see Theorem 3.30). Our exposition was mainly based on results found in [31] and [33].

In Chapter 4, we extended classic results on the hedging problem in markets with convex portfolio constraints (see [10], Chapter 5 of [44] and Chapter 9 of [23]), in the particular case of short sales prohibition, to the context of general semi-martingale financial markets (see Theorems 4.1 and 4.4). This extension was motivated by similar works in the unconstrained case by Jacka in [30] and Ansel and Stricker in [2] and is mainly based in the beautiful presentation of the optional decomposition under constraints by Föllmer and Kramkov in [22]. Along our exposition, we exhibited examples of particular types of derivatives that cannot be super-replicated or perfectly replicated in relatively general financial models (see Proposition 4.5 and Example 4.26). Related examples and results, in the Black-Scholes model and stochastic volatility models, can be found in Chapter 5 of [44] and in [11], respectively. Finally, motivated by the original work of Delbaen and Schachermayer in [14] and [15], we established an additional connection of the replication problem with the concept of maximality, properly interpreted in our context (see Theorem 4.12).

This work opens questions that motivate future research in the following directions. Delbaen and Schachermayer in [16], extended the Fundamental Theorem of Asset Pricing (FTAP) under no short sales prohibition to markets where prices are driven by non-locally bounded semi-martingales. By using the concept of σ -localization as defined in [40], one could attempt a similar extension of the FTAP under short sales prohibition presented in Chapter 1 to markets where the asset prices are not assumed to be locally bounded. It would be interesting to find examples of linear price operators satisfying no dominance, other than those obtained by taking expectation with respect to risk neutral measures (see Theorem 2.27). It is still unclear, whether one could establish necessary and sufficient conditions on the market's valuation measure under which the representation property of the underlying price process is inherited by the futures price process and whether this is always the case for models where price processes are continuous martingales with respect to at least one equivalent probability measure (see Section 3.4). Also, in this regard, it is still not completely clear whether the extreme points in $\mathcal{M}_{sup}(S)$ correspond to measures in $\mathcal{M}_{loc}(S)$ (see Theorem 3.21) and whether there is a counterpart of the Second Fundamental Theorem of Asset Pricing in markets with short sales prohibition (see Section 4.1.3). It is also unclear, whether NFLVR for a market without short sales prohibition, implies that all claims that are maximal in the sense of (i) in Theorem 4.12 are maximal in

$$\tilde{\mathcal{K}} = \{(H \cdot S)_T : H \in \tilde{A}\} \quad (5.1)$$

where \tilde{A} is the set of strategies that satisfy (i), (ii) and (iii) in Definition 2.1. Equivalently, it is unclear whether $\mathcal{M}_{loc}(S) \neq \emptyset$ and

$$\sup_{Q \in \mathcal{M}_{sup}(S)} E^Q[f] = E^{R^*}[f]$$

imply that there exists $P^* \in \mathcal{M}_{loc}(S)$ such that $E^{P^*}[f] = E^{R^*}[f]$. Also, it would be interesting to obtain a characterization of the set of claims that are maximal in \mathcal{K} (as in (5.1)) and explore whether maximality in \mathcal{K} implies maximality in $\tilde{\mathcal{K}}$. Finally, by using equilibrium arguments and taking into account liquidity considerations, one could attempt to explain the selection of the market's valuation measure and hopefully explain why the impact of short sales prohibition appears to be different in developed markets and third world economies.

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